A QUESTION ON TRACES

LOGAN HYSLOP

In this note, we investigate a problem posed by Maxime Ramzi about topological Hochschild homology. At the time of writing, this problem is listed as problem 2 under trace methods on Maxime's website under the list of mathematical questions. I learned of this question during the most recent Arbeitstagung at Bonn, when it was asked during one of the lunches, and decided to work on it the following week- what follows is the solution I came up with at that time. Without further ado, here is the precise statement of the question:

Question 1. For a ring spectrum R, we can consider the hs trace $\mathrm{K}(\mathrm{End}(R)) \to \mathrm{THH}(R)$. Is it true that every element in the image of $\mathrm{K}_0(\mathrm{End}(R)) \to \mathrm{THH}_0(R)$ is in the image of $\pi_0(R) \to \pi_0 \mathrm{THH}_0(R)$?

We answer this question in the negative. In fact, we prove:

Theorem 1. *There exists an $\mathbb{E}_\infty$-ring R such that the image of the map $\mathrm{K}_0(R) \to \mathrm{THH}_0(R)$ is not contained in the image of $\pi_0(R) \to \mathrm{THH}_0(R)$. In fact, R can be constructed out of the affine line with doubled origin.*

To be precise, recall that there is a functor from discrete rings to $\mathbb{E}_\infty$-rings taking any ring R to the Eilenberg-MacLane spectrum represented by R. This functor glues to a (contravariant) functor from schemes to $\mathbb{E}_\infty$-rings, which takes a scheme X to a ring, call it R_X, with homotopy groups $\pi_n(R_X) = H^{-n}(X, \mathcal{O}_X)$. The ring referenced in the theorem is this ring R_X for X the affine line with doubled origin (say, for simplicity, over $\mathbb{Q}$). For the remainder of this note, let's fix Y the affine line with doubled origin so that we can speak of R_Y.

To prove the theorem, we will use the following lemma:

Lemma 1. *Applying any spectra-valued localizing invariant (such as K-theory or* THH*) to the pullback square of $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} R_Y & \longrightarrow & \mathbb{Q}[t] \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{Q}[t] & \longrightarrow & \mathbb{Q}[t, t^{-1}] \end{array}$$

produces a cartesian square of spectra.

Let's first prove the theorem assuming this lemma:

Proof of Theorem 1 assuming Lemma 1. By naturality of the Dennis trace map and using Lemma 1 for K and THH, we get a commutative diagram with exact rows:

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$$\begin{array}{ccccccccc} K_1(\mathbb{Q}[t])^{\oplus 2} & \longrightarrow & K_1(\mathbb{Q}[t,t^{-1}]) & \longrightarrow & K_0(R_Y) & \longrightarrow & K_0(\mathbb{Q}[t])^{\oplus 2} & \longrightarrow & K_0(\mathbb{Q}[t,t^{-1}]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{THH}_1(\mathbb{Q}[t])^{\oplus 2} & \longrightarrow & \mathrm{THH}_1(\mathbb{Q}[t,t^{-1}]) & \longrightarrow & \mathrm{THH}_0(R_Y) & \longrightarrow & \mathrm{THH}_0(\mathbb{Q}[t])^{\oplus 2} & \longrightarrow & \mathrm{THH}_0(\mathbb{Q}[t,t^{-1}]). \end{array}$$

Now, we know that $K_1(\mathbb{Q}[t]) \simeq K_1(\mathbb{Q}) \simeq \mathbb{Q}^\times$, and $K_1(\mathbb{Q}[t,t^{-1}]) \simeq \mathbb{Q}[t,t^{-1}]^\times \simeq \mathbb{Q}^\times \times \mathbb{Z}$. Since we assumed everything above was rational, THH is equivalent to the Hochschild homology over $\mathbb{Q}$, so by HKR, $\mathrm{THH}_1(\mathbb{Q}[t]) \simeq \mathbb{Q}[t]dt$, and $\mathrm{THH}_1(\mathbb{Q}[t,t^{-1}]) \simeq \mathbb{Q}[t,t^{-1}]dt$. Using the long exact sequence, we find that $\mathrm{THH}_0(R_Y) \simeq \mathbb{Q}[t] \oplus \mathbb{Q}[t,t^{-1}]/\mathbb{Q}[t]$, and the image of $\pi_0(R_Y) \to \mathrm{THH}_0(R_Y)$ is the $\mathbb{Q}[t]$ summand. Now, we look at $t^{-1} \in K_1(\mathbb{Q}[t,t^{-1}])$, and noting this comes from $H_1^{grp}(\mathrm{GL}_1(\mathbb{Q}[t,t^{-1}])) \to \mathrm{HH}(\mathbb{Q}[t,t^{-1}]/\mathbb{Q})$, we can determine this map explicitly: t^{-1} corresponds to the element $[t^{-1} : t]$, which maps to $t^{-1}dt$ under the HKR isomorphism. In particular, the image of t^{-1} under $K_1(\mathbb{Q}[t,t^{-1}]) \to \mathrm{THH}_1(\mathbb{Q}[t,t^{-1}]) \to \mathrm{THH}_0(R_Y)$ is $t^{-1}dt \in \mathbb{Q}[t,t^{-1}]/\mathbb{Q}[t]dt$, which is not in the image of the map $\pi_0(R_Y) \to \mathrm{THH}_0(R_Y)$, yielding the desired contradiction by considering the image of t^{-1} in $K_0(R_Y)$. ☐

Now, for the proof of Lemma 1. We have to pass to the categorical level. We first recall that there is a (in fact fpqc) sheaf of (stably symmetric monoidal) ∞-categories on schemes taking a scheme X to (derived) quasicoherent sheaves on X, $\mathcal{D}(\mathrm{QCoh}(X))$, glued from $R \mapsto \mathcal{D}(R)$. In particular, since localizations of schemes map to localizations on the categorical level, we get that any pushout square along open immersions (corresponding to gluing schemes) maps to a pullback square under a localizing invariant. Thus, to prove Theorem 1, it suffices to show that quasicoherent sheaves over Y are equivalent to modules over R_Y. This follows from a general claim:

Lemma 2. *Suppose X is a scheme with a cover by finitely many affine opens with affine intersection such that any (underived) quasicoherent sheaf on X with vanishing global sections is equivalent to zero. Then we have an equivalence of stably symmetric monoidal ∞-categories $\mathcal{D}(\mathrm{QCoh}(X)) \simeq R_X - \mathrm{Mod}$.*

To fix some notation in the proof, hom will refer to the mapping spectrum, and Hom to the mapping space.

Proof. We recover this from the Schwede-Shipley Theorem (see Higher Algebra 7.1.2.7 for the precise statement used). The hypotheses on X ensure that taking cohomology commutes with filtered colimits so that the unit of $\mathcal{D}(\mathrm{QCoh}(X))$ is compact, which reduces us to the claim that the unit generates $\mathcal{D}(\mathrm{QCoh}(X))$. Fix our chosen finite affine cover $U_1, ..., U_n$ of X. For this, take any $M \in \mathcal{D}(\mathrm{QCoh}(X))$ such that $\mathrm{hom}(R_X, M) \simeq 0$. By the assumption on X, we can put a t-structure on $\mathcal{D}(\mathrm{QCoh}(X))$ by gluing the t-structures on the derived categories for the U_i. This t-structure allows us to identify the heart with the usual category of quasicoherent sheaves on X. Now, if M is not equivalent to zero, we can find some n so that $\tau_{\geq n} M \neq 0$, and up to shifts, we may assume $n = 0$. Now, the unit is connective by definition of the t-structure, so $\mathrm{Hom}(R_X, M) \simeq \mathrm{Hom}(R_X, \tau_{\geq 0} M)$. Up to possibly shifting M again, we may assume that $\pi_0^\heartsuit M \neq 0$. Now, we can examine the cofiber sequence $\pi_0^\heartsuit M \to \tau_{\geq 0} M \to \tau_{\geq 1} M$. By hypothesis, $\pi_0^\heartsuit M \neq 0$, implying $\pi_0 \mathrm{hom}(R_X, \pi_0^\heartsuit M) \neq 0$, so either $\pi_0 \mathrm{hom}(R_X, \tau_{\geq 0} M) \neq 0$ or

$\pi_1 \hom(R_X, \tau_{\geq 1} M) \neq 0$, by the associated long exact sequences. In the former case, we have a contradiction, since $\pi_0 \hom(R_X, \tau_{\geq 0} M) \simeq \pi_0 \hom(R_X, M)$, and in the latter case, we similarly reach a contradiction since $\pi_1 \hom(R_X, \tau_{\geq 1} M) \simeq \pi_1 \hom(R_X, M)$. Thus, if $\hom(R_X, M) \simeq 0$, $M \simeq 0$, i.e., R_X generated $\mathcal{D}(\mathrm{QCoh}(X))$, and thus by Schwede-Shipley, we get the claimed equivalence. □

Now, with this in hand, it is easy to show:

Proof of Lemma 1. By Lemma 2, it suffices to show that every quasicoherent sheaf on the affine line with doubled origin with vanishing global sections is identically zero. Indeed, then $\mathcal{D}(\mathrm{QCoh}(Y)) \simeq R_Y - \mathrm{Mod}$, and thus the pullback square in the statement of lemma 1 maps to a pullback square of stable ∞-categories upon passing to modules, with all functors being Verdier localizations. So, let M be a nonzero quasicoherent sheaf on the affine line with doubled origin, and let M_1, M_2 be the restrictions to the two affine lines. Setting up the Cech complex, it looks like $M_1 \oplus M_2 \to M_1[t^{-1}]$, where we have used $M_2[t^{-1}] \simeq M_1[t^{-1}]$ implicitly. Now, if $M_1[t^{-1}] \neq 0$, then since $M_2[t^{-1}] = M_1[t^{-1}]$, we can take any $m_1 \in M_1$ with nonzero image in $M_1[t^{-1}]$, and write $m_1 = m_2 t^{-n}$ for some $m_2 \in M_2$. But then $(t^n m_1, -m_2) \in M_1 \oplus M_2$ is a nonzero element in the kernel of the map in our complex, contributing a nonzero global section to M. On the other hand, if $M_1[t^{-1}] = 0$, then by hypothesis, either M_1 or M_2 is nonzero, but then these contribute nonzero global sections to M since the Cech complex in this case just becomes $M_1 \oplus M_2 \to 0$, whence the claim. □

What was left out of the question above is the interpretation of what it means, per Maxime's phrasing "i.e., is the Hattori-Stallings trace of any endomorphism of some perfect R-module equal to the trace of some endomorphism of R?" What we have shown above is that this is not even true for the trace of the identity endomorphism of a perfect R-module. One may ask what this module is, and we can explicitly describe it. Namely, we note that the picard group of Y is nontrivial, coming from the line bundle L glued from the isomorphism $\mathbb{Q}[t, t^{-1}] \xrightarrow{t} \mathbb{Q}[t, t^{-1}]$. We have a map from the Picard group to K_0, which takes a line bundle L to $1 - [L]$. For our line L, this is witnessed by the complex $\Sigma L \oplus R_Y$. This maps to zero in the K_0 of each of the $\mathbb{Q}[t]$s, so lives in the span of the t^{-1} summand of $K_0(R_Y) \simeq \mathbb{Z} \times \mathbb{Z}$, giving a perfect R_Y module such that the trace of the identity endomorphism does not arise as the trace of an endomorphism of R_Y, as desired.