MODULAR FORMS, MANIFOLDS AND MORE

LOGAN HYSLOP

Abstract. In this final project, we will one way in which the role of modular forms plays out in differential topology: elliptic genera. §1 will begin our tale with a discussion on genera and what it means to be an elliptic genus classically. Following this, §2 will introduce the relevant facts about modular forms that we will use throughout the paper. In §3, we discover how to construct an elliptic genus out of an elliptic curve with a chosen 2-torsion point, and how this perspective leads us to consider "elliptic genera" as specializations of one elliptic genus taking values in the ring $\mathcal{M}_*(\Gamma_0(2))$ of modular forms with level $\Gamma_0(2)$, and how important genera are recovered by taking values at the cusps. In the last section, we briefly touch on the connections between elliptic genera and some advanced topics that we will not have time to cover in any detail.

Contents

§1. Recollections on Genera

Throughout this paper, all of our manifolds are implicitly assumed to be smooth unless otherwise stated.
Our story begins with Thom's definition of a cobordism between smooth manifolds. In this case, we consider the ring of cobordism classes of closed, oriented, smooth manifolds up to oriented cobordism. Explicitly, this is a graded ring, with underlying graded abelian group $\Omega_n = \{\text{closed, oriented } n-\text{dimensional manifolds } M\}/\sim$, where we say that $M\tilde{N}$ if there is a compact, oriented $n+1$-dimensional manifold with boundary W such that $\partial W \simeq M \coprod -N$, where $-N$ denotes N with the reversed orientation. This can be made into an abelian group with operation $M + N := M \coprod N$[1], and the full collection inherits a graded-commutative ring structure where $[M]\cdot[N] := [M \times N]$. We now have:

Definition 1.1. The **oriented cobordism ring** Ω_*^{or} is the graded commutative ring as described above. For the purposes of this paper, we adopt the (non-standard) notation $\Omega_* := \Omega_*^{or} \otimes \mathbb{Q}$, and will simply write Ω when considering Ω_* without reference to the graded structure.

For the majority of the paper, we will work with Ω_*, due to the following theorem of Thom:

Theorem 1.2 (Thom)**.** *The ring Ω_* is isomorphic as a graded ring to a polynomial algebra $\mathbb{Q}[x_{4n}]$ with a generator in each degree $4n$ $n > 0$. This isomorphism may be chosen in such a way that x_{4n} is represented by the cobordism class of the complex projective space $\mathbb{P}^{2n}(\mathbb{C})$.*

[1] One may allow $\emptyset$ to be an n-dimensional manifold to get a canonical zero element, otherwise $M \coprod -M$ for any M can serve as a zero element.

□

With this in hand, we can now define:

Definition 1.3. A(n) (R-valued) **genus** is a ring homomorphism $\varphi : \Omega_* \to R$ for some $\mathbb{Q}$-algebra R, which we require to be a domain.

The notion of a genus is very closely connected to the theory of characteristic classes, and more specifically Pontrjagin classes. To be precise, given an even power series $Q(x) = 1 + a_2 x^2 + \ldots \in R[[x^2]]$, and a $4n$-dimensional oriented closed manifold M, we can form the element $\int_M Q(x_1)\ldots Q(x_n) \in R$, where $x_1^2, \ldots, x_n^2 \in H^4(M, \mathbb{Q})$ are the Pontrjagin roots of TM[2], which we will define as $\varphi_Q(M)$, and we will set $\varphi_Q(M) = 0$ if $\dim M \not\equiv 0 \mod 4$. One can show as in [Hir+13] that this defines a genus, and setting $f = \frac{x}{Q(x)} \in R[[x]]$, g the compositional inverse to f, that

Lemma 1.4. $g'(x) = \sum_{n\geq 0} \varphi_Q(\mathbb{P}^n(\mathbb{C}))x^n$.

These power series arise naturally by considering virtual submanifolds of M, which gives rise to a formal group law for the genus, g' arising as its logarithm, but for the sake of brevity we forgo the formal discussion, referring instead to [Hir+13] for a more detailed treatment.

We now mention a couple notable examples of ($\mathbb{C}$-valued) genera that will follow us later:

Example 1.5. (1) **The $\hat{A}$-genus** is the genus defined by $\hat{A}(M) := \varphi_Q(M)$ for $Q(x) = \frac{x/2}{\sinh(x/2)}$. In this case, $f(x) = 2\sinh(x/2)$ satisfies the differential equation $(f')^2 = 1 + \frac{1}{4}f^2$, and then $g'(x)^2 = \frac{1}{1+\frac{1}{4}x^2} = \sum_{n\geq 0}(-1)^n(\frac{x}{2})^{2n}$ tells us the value of the $\hat{A}$-genus on complex projective spaces by inductively constructing the square root g' uniquely determined by the fact that $g'(0) = 1$.
(2) **The L-genus** is the genus defined by $L(M) := \varphi_Q(M)$ when $Q(x) = \frac{x}{\tanh(x)}$. In this case, we can compute that $(f')^2 = 1 - 2f^2 + f^4$, so $g'(x) = \frac{1}{1-x^2} = \sum_{n\geq 0} x^{2n}$, so that the L-genus takes value 1 on all $\mathbb{P}^{2n}(\mathbb{C})$.

The L-genus has an equivalent description as the signature of a manifold. The signature $\mathrm{sign}(M)$ of a closed, oriented $4k$-dimensional manifold M is the signature of the bilinear form $\langle x, y\rangle := \int_M x \wedge y$ on the middle dimensional cohomology $H^{2k}_{dR}(M)$. The fact that $\mathrm{sign}(M) = L(M)$ follows from the Hirzebruch signature theorem, which can be proven following Hirzebruch's original proof or by applying the Atiyah-Singer index theorem to an appropriate elliptic operator.

One can examine the power series $h(x) := \sum_{n\geq 0} \varphi(\mathbb{P}^n(\mathbb{H}))x^{2n}$ with coefficients the value of our genus on quaternionic projective spaces. As in [Hir+13], we can show that $h(f(x)) = \frac{f(2x)}{2f(x)f'(x)}$. This leads one to the following definition:

Definition 1.6. The genus associated to f is said to be **elliptic** if $f(2x) = \frac{2f(x)f'(x)}{1-\varepsilon f^4}$.

If we specify that f is an odd power series such that $g'(x) = 1 + \delta x^2 + \ldots$, then the data of ε and δ uniquely determine f, which is quickly verified by examining the coefficients above in $f(2x) = 2f(x)f'(x)(\sum_{n\geq 0} f^{4n}(x))$. For the x^3 coefficient, this can be freely chosen (for the

[2] We are lying a little bit here, since these may not literally exist in $H^4(M, \mathbb{Q})$, but one can use the splitting principle to pass to a case when they actually do exist to make the definition. Using the fact $Q(x_1)...Q(x_n)$ has homogeneous components given by symmetric polynomials in the $x_1^2, ..., x_n^2$ in each dimension, in particular in dimension $4n$, one can make sense of this as long as one knows the total Pontrjagin class of TM.

RHS, we have $(x + a_3x^3 + \ldots)(1 + 3a_3x^2 + \ldots)(1 + \ldots) = x + 4a_3x^3 + \ldots)$, and is where the choice of δ is needed, but for $n > 3$, we get a formula for the coefficient of x^n in terms of the lower degree coefficients, giving uniqueness. For an odd power series f, this is equivalent to the condition that $(f')^2 = 1 - 2\delta f^2 + \varepsilon f^4$, which can be seen by proving a similar uniqueness claim here, then following the strategy of proof on page 27 of [Hir+13].

Example 1.7. The L-genus and $\hat{A}$-genus in particular were both elliptic genera, giving our first examples. For the L-genus, $\varepsilon = \delta = 1$, and for the $\hat{A}$-genus, $\varepsilon = 0$, $\delta = \frac{-1}{8}$.

§2. A Digression on Modular Forms

In this section, we collect some of the results from the general theory of modular forms to be used in this paper. For the most part, these were all discussed in class. As a first goal, we wish to determine the ring $\mathcal{M}_*(\Gamma_0(2))$. To start with, note:

Lemma 2.1. *The modular curve $X_0(2)$ has genus 0, 2 cusps, 1 elliptic point of order 2, and no elliptic points of order 3.*

Proof. By [Woh64] the lcm of the index of the cusps for a congruence subgroup Γ is the smallest N such that $\Gamma(N) \subseteq \Gamma$. In particular, the least common multiple of the cusps for $\Gamma_0(2)$ must be 2. Since $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(2)] = 3$, there are at most 3 cusps, and since at least one of them has to have index 2, there can only be two cusps, one of index 2 and one of index 1. We can even be explicit, noting that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \Gamma_0(2)$, so that the cusp ∞ has index 1, and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \notin \Gamma_0(2)$, but $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \in \Gamma_0(2)$, so the cusp 0 has index 2. Plugging this into our genus formula: $g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2} = 1 + \frac{3}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - 1$, and using that the genus is nonnegative, we find that we must have $g = 0$, $\varepsilon_2 = 1$, $\varepsilon_3 = 0$, as claimed. □

With this in hand, we can use Diamond and Shurman Theorem 3.5.1 to find that $\dim \mathcal{M}_k(\Gamma_0(2)) = (k-1)(-1) + \lfloor \frac{k}{4} \rfloor + \frac{k}{2} \cdot 2 = 1 + \lfloor \frac{k}{4} \rfloor$ for $k \geq 2$ even (and $-id \in \Gamma_0(2)$, so $\mathcal{M}_{2k+1}(\Gamma_0(2)) = 0$). If we had a graded polynomial algebra $\mathbb{C}[x, y]$ with $|x| = 2$, $|y| = 4$, with an injective map to $\mathcal{M}_*(\Gamma_0(2))$, comparing their Poincaré series would show that they are isomorphic. To this end, will need the notion of an index 0 Jacobi form:

Definition 2.2. A **Jacobi form of index 0, weight k and level Γ** is a function $\Phi(\tau, z) : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ such that $\Phi(\tau, -)$ is elliptic for the lattice $2\pi i(\tau\mathbb{Z} + \mathbb{Z})$, and $\Phi(\gamma(\tau), xj(\gamma, \tau)^{-1}) j(\varphi, \tau)^{-k} = \Phi(\tau, z)$.

Here $\mathcal{H}$ denotes the upper half plane. The only index 0 Jacobi form we will actually work with is the weight 2 level $\mathrm{SL}_2(\mathbb{Z})$ Jacobi form given by the familiar Weierstrass $\wp$-function (following Hirzebruch's non-standard choice of lattice points)

$$\wp(\tau, z) = \frac{1}{z^2} + \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(z - 2\pi i(m\tau + n))^2} - \frac{1}{(2\pi i(m\tau + n))^2}.$$

We have the following important property of index 0 Jacobi forms, which can be proven via elementary complex analysis:

Theorem 2.3 ([Hir+13] Theorem I.3.1)**.** *If Φ is a Jacobi form as above of index 0, weight k and level Γ, $(\alpha, \beta) \in \mathbb{R}^2$, and $g_n(\tau)$ the nth coefficient in the Taylor expansion of $\Phi(\tau, -)$ at $2\pi i(a\tau + \beta)$, then $g_n(\tau)|_{[\gamma]_{n+k}} = g_n$ for all $\gamma \in \Gamma$ such that $(\alpha, \beta)\gamma \equiv (\alpha, \beta) \mod \mathbb{Z}^2$. Furthermore, if one denotes this g_n now by $g_n^{(\alpha,\beta)}$, then we have that $g_n^{(\alpha,\beta)}(\tau)|_{[\gamma]_{n+k}} = g_n^{(\alpha,\beta)\gamma}(\tau)$, and $g_n^{(\alpha,\beta)}$ depends only on $(\alpha, \beta) \mod \mathbb{Z}^2$.*

Remark 2.4. This should be seen as a natural generalization of the fact that the nth Taylor coefficient of $\wp(\tau, -)$, our familiar Eisenstein series, form modular forms of weight n for the whole group $\mathrm{SL}_2(\mathbb{Z})$.

Proof. The first statement is proved in [Hir+13], so we prove the more general claim, which follows a similar method. Note that $g_n^{(\alpha,\beta)} = \frac{1}{2\pi i}\oint \frac{\Phi(\tau, z+2\pi i(\alpha\tau+\beta))}{z^{n+1}}dz$, so we have that:

$$\begin{aligned}
g_n^{(\alpha,\beta)}|_{[\gamma]_{n+k}} &= \frac{1}{2\pi i}\oint \frac{\Phi(\gamma\tau, z+2\pi i(\alpha\gamma(\tau)+\beta))}{z^{n+1}} j(\gamma,\tau)^{-n-k}dz \\
&= \frac{1}{2\pi i}\oint \frac{\Phi(\gamma\tau, (zj(\gamma,\tau)+2\pi i(\alpha'\tau+\beta'))j(\gamma,\tau)^{-1})}{(zj(\gamma,\tau))^{n+1}} j(\gamma,\tau)^{-k}d(zj(\gamma,\tau)) \\
&= \frac{1}{2\pi i}\oint \frac{\Phi(\tau, z+2\pi i(\alpha'\tau+\beta'))}{z^{n+1}}dz \\
&= g_n^{(\alpha',\beta')}
\end{aligned}$$

where $(\alpha',\beta') = (\alpha,\beta)\gamma$, as claimed. ◻

From this, we make a few definitions. Define $e_1(\tau) := \wp(\tau,\pi i)$, $e_2(\tau) := \wp(\tau,\tau\pi i)$, and $e_3(\tau) := \wp(\tau,\tau\pi i+\pi i)$. By Theorem 2.3, we have that e_1 is a modular form[3] on the subgroup of all γ with $(0,\frac{1}{2})\gamma \equiv (0,\frac{1}{2}) \mod \mathbb{Z}^2$, which we can see imposes the requirement, with $\gamma = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$, $c \equiv 0 \mod 2$, so that e_1 is a modular form of weight 2 and level $\Gamma_0(2)$. Similarly, e_2 is a modular form of weight 2 and level $\Gamma^0(2)$, and e_3 is a modular form of weight 2 and level $\{X \in \mathrm{SL}_2(\mathbb{Z}) : X \equiv id \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix} \mod 2\}$. Furthermore, $|_{[\gamma]_2}$ permutes the e_i for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

We now define some modular forms that will be of great importance later: let $\delta(\tau) := \frac{-3}{2}e_1(\tau) \in \mathcal{M}_2(\Gamma_0(2))$, and $\varepsilon(\tau) := (e_1-e_2)(e_1-e_3) \in \mathcal{M}_4(\Gamma_0(2))$. These are holomorphic on $\mathcal{H}$, and one can see that they are also holomorphic at cusps We have the following theorem:

Theorem 2.5. $\mathcal{M}_*(\Gamma_0(2)) = \mathbb{C}[\delta,\varepsilon]$.

Proof. By our remarks above, it suffices to show that δ and ε do not satisfy any nontrivial relations. As usual, it suffices to find a point e where δ vanishes and ε does not (or vice versa), as then any minimal relation $\sum_i a_i\delta^{2i}\varepsilon^{n-i}$ must have $a_0 = 0$, and we can divide out δ^2 to get a smaller relation, a contradiction. Hirzebruch proves by means of a valence formula that taking e to be the elliptic point of order 2 works. We can also do this by examining cusps. Note that $e_1(\tau) = \frac{-1}{\pi^2}(1+\sum \frac{1}{(1-2(m\tau+n))^2} - \frac{1}{(2m\tau+2n)^2})$, and taking $\tau \longrightarrow +i\infty$, playing a bit fast and loose with limits reduces us to just $\frac{-1}{\pi^2}(1+\sum_{n\neq 0}\frac{1}{(2n-1)^2} - \frac{1}{(2n)^2}) = \frac{-2}{\pi^2}(\frac{\pi^2}{12}) = \frac{-1}{6}$. For e_2 and e_3, it is even simpler, since there we have $z = \pi i\tau$ resp. $\pi i\tau + \pi i$, which grow large as τ does, leaving us with only $\sum_{n\neq 0}\frac{1}{\pi^2(2n)^2} = \frac{1}{2\pi^2}\sum_{n\geq 1}\frac{1}{n^2} = \frac{1}{12}$. To calculate the value of δ at the cusp 0, we need to do, for $\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$, $\delta|_{[\alpha]_2}(\infty) = -\frac{3}{2}e_1|_{[\alpha]_2}(\infty) = \frac{-3}{2}e_2(\infty) = -\frac{1}{8}$. For ε at this cusp, we note that $(e_1-e_3)|_{[\alpha]_2} = (e_2-e_3)$, and $e_2(\infty) = e_3(\infty) = \frac{1}{12}$, so that ε takes the value 0 at the other cusp, and δ is nonzero at this cusp. ◻

The above is a reflection of the underlying geometric philosophy at play. These e_i, for a fixed τ, are the images of the 2-torsion points of the elliptic curve under the Weierstrass function for the chosen lattice for the curve. The same definitions work for $e_i \in \mathbb{C}$ for an arbitrary lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with fixed 2-torsion point $\omega_1/2$ (and changing to a standard basis will only possibly

[3] Or at least weakly modular, we will look at the behavior of the e_i at ∞ shortly which will give that it is a modular form.

change the associated genus up to a normalization factor). The above cases correspond to the degeneration $\omega_1 \to 0$, which makes e_2 tend to e_3, and so $\varepsilon = \delta^2$, in the case of the cusp at ∞, and the other two cases, we could have either $\omega_1 \to \omega_2$, or $\omega_2 \to 0$, in which case $e_1 = e_2$ or $e_1 = e_3$, giving $\varepsilon = 0$, exactly the second case seen above.

§3. Modular Forms and Elliptic Genera

Given a lattice Λ in the plane $\mathbb{C}$, there is a way to construct a ($\mathbb{C}$-valued) elliptic genus out of it, by using the Weierstrass $\wp$-function for the lattice. On the other hand, we can construct all of these genera at once as a $\mathcal{M}_*(\Gamma_0(2))$-valued elliptic genus, taking for the δ, ε mentioned in §1 exactly the modular forms from §2 with these names. We make the following definition:

Definition 3.1. Define the **universal elliptic genus** as the elliptic genus $\varphi_{\mathcal{E}\ell\ell} : \Omega \to \mathcal{M}_*(\Gamma_0(2)) = \mathbb{C}[\delta, \varepsilon]$ associated to the odd power series $f(x) = x + \ldots$ with coefficients in $\mathbb{C}[\delta, \varepsilon]$ which solves $(f')^2 = 1 - 2\delta f^2 + \varepsilon f^4$ as in §1.

Any other elliptic genus over a $\mathbb{C}$-algebra (the correct notion above is really to work with rational modular forms to get the truly universal case) can of course be obtained from this one by the map $\mathbb{C}[\delta, \varepsilon] \to R$ picking out the elements of the synonymous names. One benefit of thinking about this universal target ring as the ring $\mathcal{M}_*(\Gamma_0(2))$ is that our elliptic genus now assigns an actual modular form $\varphi_{\mathcal{E}\ell\ell}$ to any oriented closed smooth manifold M, and if M has dimension $4k$, $\varphi_{\mathcal{E}\ell\ell}(M)$ has weight $2k$. A natural question to ask is what do the specializations to the cusps look like?

Proposition 3.2. *At the cusp ∞, we have that $\varphi_{\mathcal{E}\ell\ell}|_\infty$ is, up to a normalization factor, the L-genus, and at the other cusp 0, $\varphi_{\mathcal{E}\ell\ell}|_0$ is the $\hat{A}$-genus.*

Proof. This follows from the fact that $\delta(\infty) = \frac{-3}{2} \cdot \frac{-1}{6} = \frac{1}{4}$, so that $\varphi_\infty(\mathbb{P}^2(\mathbb{C})) = \frac{1}{4}$, and $\varepsilon(\infty) = (\frac{-1}{6} - \frac{1}{12})^2 = \frac{1}{16}$. The case $\varepsilon = \delta^2$ is, up to normalization, the L-genus, where the normalization we use is changing $f(x)$ to $\frac{f(2x)}{2}$, which amounts to $L(M) = 4^{\dim(M)/4}\varphi_\infty(M)$.

At the other cusp, letting $\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ again, we had $\delta|_{[\alpha]_2}(\infty) = \frac{-3}{2} \cdot \frac{1}{12} = \frac{-1}{8}$, and $\varepsilon|_{[\alpha]_2}(\infty) = 0$, which was exactly the case of the $\hat{A}$-genus, without any need to normalize. □

So, we see that two of the most important examples of elliptic genera- the signature and the $\hat{A}$-genus- arise from the values on the cusps of the universal elliptic genus. The other point that we could ask about is the elliptic point on $X_0(2)$, where, by the valence formula in [Hir+13], δ vanishes but ε does not. Thus, up to normalization, the elliptic genus specified at this point has $\delta = 0$ and $\varepsilon = 1$. It is determined by a power series $f(x) = x + a_1x^3 + \ldots$ with $a_1 = 0$, satisfying $(f')^2 = 1 + f^4$. If we plug this into Wolfram alpha, it tells us a solution in terms of the Jacobi elliptic function sn. I do not know if this actually has a name as a genus.

§4. Outlook

In this section we collect a general overview of some results in elliptic genera that will not be elaborated on due to lack of time and space. One key point that I'd love to come back to and add more detail about later is the connection between elliptic genera and the free loop space $\mathcal{L}(X)$ of a manifold X. The theory of genera is large and broad, with applications permeating throughout modern mathematics, and a paper as short as this could not hope to do the full theory justice. As recompense, here are a few other connections that are discussed in [Hir+13] or elsewhere:

§4.1. Connections to Moonshine. One can define the so-called Witten genus φ_W defined from $Q(x) = \frac{x/2}{\sinh(x/2)} \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})}$, which takes values in the ring of power series $[\quad[q]]$, and if we restrict to manifolds admitting a spin structure, it lands in $\mathbb{Z}[[q]]$. There is a certain characteristic class, the vanishing of which on M ensures that $\varphi_W(M)$ is the q-expansion of a modular form. Hirzebruch-Berger-Jung relate the Witten genus to values of the $\hat{A}$-genus with coefficients, and proves that for a 24-dimensional manifold X with $\hat{A}(X) \neq 0$, there is a formula relating values of $\hat{A}$ with coefficients in various vector bundles and the j-function. In particular, there is a question posed as to whether there exists a 24-dimensional manifold with prescribed $\hat{A}$-genus and certain cohomology classes vanishing, which would have a chance for the monster group to act on it via diffeomorphisms, providing a way to construct many representations of this group.

§4.2. Elliptic Cohomology. Since elliptic genera provide a way to obtain a topological invariant out of an elliptic curve, it is a natural question to ask whether this can be improved to a full cohomology theory. The theory of elliptic cohomology has progressed rapidly and found uses in the study of chromatic homotopy theory and physics. We will focus only on the pre-history, and not the modern treatment. There are several closely related notions of an orientation of a cohomology theory.

Digression: complex-oriented cohomology theoires: A **complex orientation** on a multiplicative cohomology theory h^* is a class $u \in h^2(\mathbb{CP}^\infty)$ which restricts along $i : \mathbb{CP}^1 \to \mathbb{CP}^\infty$ to $i^*(u) \in h^2(\mathbb{CP}^1) \simeq h^0(*)$ to the identity $1 \in h^0(pt)$. Complex oriented cohomology theories are the foundation of modern chromatic homotopy theory, since they form a "nice" family of cohomology theories, each giving rise to formal group laws. Quillen showed that the complex cobordism ring (defined similarly to the oriented cobordism ring from the start but with stably almost complex manifolds instead) is isomorphic to the Lazard ring, the ring with a universal formal group law. Further work in the field from the likes of Hopkins, Ravenel, Landweber and others, proved that one can go the other way, that suitably nice formal group laws give rise to complex-oriented cohomology theories, and one can even make a stratification of the category of spectra in terms of the moduli stack of formal group laws, which is what is now known as chromatic homotopy theory.

Since we are working with oriented cobordisms, we should ask for a real orientation. The general strategy for this is to abuse complex conjugation, so we define **an orientation** of a multiplicative homology theory h^* (on which we will always assume 2 is inverted from now on so that in particular there isn't any funny business with homotopy fixed points[4]) with a complex orientation $u \in h^2(\mathbb{CP}^\infty)$ such that $\bar{u} = -u$ where $\bar{u}$ denotes the image under the action of complex conjugation. This complex orientation gives rise to a formal group law F

[4] basically group cohomology for C_2

which corresponds to the manner one would build a formal group law out of $f(x)$ for a genus, characterized by $F(u,v) = f(g(u) + g(v))$, with notation as in §1. We are identifying these as such, since a real orientation gives rise to a map $MSO[\frac{1}{2}] \to h^*$, which, applied to a point, gives a ring map $\varphi : \Omega^*(*) \to h^*(*)$, providing us with a genus. One of the initial motivations for elliptic cohomology is the following example:

Example 4.1. Consider the ring $\mathbb{C}[\delta, \varepsilon]$ as a differential graded $\mathbb{C}$-algebra with trivial differentials, where we put δ in (homological) degree 4 (cohomological degree -4) and ε in homological degree 8 (cohomological degree -8). This forms a rational cohomology theory, and one can choose an orientation for it as in [Seg87] (2.3) in such a way that the induced map recovers the universal elliptic genus $\varphi_{\mathcal{E}\ell\ell}$ from before.

References

[Hir+13] F. Hirzebruch, P.S.T. Landweber, T. Berger, and R. Jung. Manifolds and Modular Forms. Aspects of Mathematics. Vieweg+Teubner Verlag, 2013. ISBN: 9783663107262. URL: https://books.google.com/books?id=0j3vCAAAQBAJ.

[Seg87] Graeme Segal. "Elliptic cohomology". eng. In: Séminaire Bourbaki 30 (1987-1988), pp. 187–201. URL: http://eudml.org/doc/110095.

[Woh64] Klaus Wohlfahrt. "An extension of F. Klein's level concept". en. In: Illinois J. Math. 8.4 (1964), pp. 529–535. URL: http://dml.mathdoc.fr/item/1256059574.