OVERCONVERGENT MODULAR FORMS AND MODULAR CURVES AT INFINITE LEVEL

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Abstract. In this paper, we explain Scholze's construction of a Perfectoid space $X(p^\infty)$ which gives the "modular curve at infinite level." We will take the approach of Lurie in [Lur20], and explain his construction in §2. In §3, we construct an analytic adic space $X(N)[[T]]$ fibered over $\mathrm{Spa}(\mathbb{Z}_p[[T]])$, which acts as a sort of adic deformation of the overconvergent modular curve. In particular, the fiber of $X(N)[[T]]$ over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \to \mathrm{Spa}(\mathbb{Z}_p[[T]])$ $T \mapsto p^r$ for $r \geq 1$, gives rise to the rigid space X_r we defined in class. In particular, one can define $\mathcal{M}_k^{def}(N) := H^0(X(N)[[T]], \omega^{\otimes k})$, which admits maps to each $\mathcal{M}_k(N, r)$.

Contents

§1. Introduction/Recollections

This final project, we are trying something new. §2 will provide a brief exposition of a construction of Lurie [Lur20], although Lurie's paper is very good on its own so this will be a bit brief. In §3, the story will be a bit less well-refined, since the deformations considered (to the best of my knowledge) have not appeared in the literature, and were more of a fun way to play around with some of the ideas. Although we do hope these deformations could be useful in some way, due to time constraints on writing this final project, we will not be able to prove anything nontrivial with them, and there is still even some ambiguity between different definitions about what the right notions should be.

We freely use the language of adic spaces throughout this paper, referring to [SW20] for a formal introduction. Of particular focus will be analytic adic spaces. Recall that a point $x \in \mathrm{Spa}(A, A^+)$ is analytic if $\ker(x)$ is not open, and an adic space is analytic if all its points are analytic. These are the spaces most closely related to the classical theory of rigid spaces. Below, we will often deal with the map taking a formal scheme to an adic space, the construction of which is detailed in [SW13], section 2.2. The rigid generic fiber can usually be considered as an adic space arising from the fiber over the generic point $\eta \in \mathrm{Spa}(\mathbb{Z}_p)$ (corresponding to $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$), which will relate the constructions made here back to the more classical constructions detailed in class.

Conventions. We will often say moduli space below which, if there is some $N \geq 5$ prime to p is involved, will often involve an actual space, although, especially if we are in a case where we can specialize to $p = 0$, if we do not have this, we may be working with a moduli stack instead which we have abusively called a space. $\mathbb{Q}_p^{cyc}$ will denote the completion of $\mathbb{Q}_p[\zeta_{p^\infty}]$, and $\mathbb{Z}_p^{cyc}$ its ring of integers.

§2. Adic Modular Curves and Modular Curves at Infinite Level

First, recall how to construct an adic space (or adic stack in some cases) from the moduli stack $\overline{\mathcal{E}\ell\ell}(N)$ of generalized elliptic curves with full level N structure, considered as a DM $\mathbb{Z}_p$-stack. We can get from this a sheaf of groupoids on the category $\text{Nilp}_{\mathbb{Z}_p}^{op}$, where $\text{Nilp}_{\mathbb{Z}_p}$ is the category of commutative $\mathbb{Z}_p$-algebras on which p is nilpotent. We can construct, from any such "formal stack"[1], an adic stack $\overline{\mathcal{E}\ell\ell}(N)^{ad}$ over $\text{Spa}(\mathbb{Z}_p)$ as in [SW13] section 2.2, in particular:

Definition 2.1. Let $\mathcal{X}$ be a formal stack over $\mathbb{Z}_p$, locally admitting a finitely generated ideal of definition. There is an adic stack $\mathcal{X}^{ad}$ which takes an affinoid adic space $\text{Spa}(A, A^+)$ to the sheafification of $(A, A^+) \mapsto \varinjlim_{A_0 \subseteq A^+} \mathcal{X}(A_0)$, where the limit is over the open bounded subalgebras of A^+. Here the limit must be interpreted as a 2-categorical limit in case $\mathcal{X}$ does not come from a formal algebraic space.

Using this, we get adic stacks $\overline{\mathcal{E}\ell\ell}(N)^{ad}$ over $\text{Spa}(\mathbb{Z}_p)$, and taking the adic generic fiber over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ gives an analytic adic space (at least for $N \geq 5$) which recovers Katz's rigid space $X^{rig}(N)$.

We have an inverse system of adic spaces (stacks if $N < 5$) given by $\ldots \to \overline{\mathcal{E}\ell\ell}(Np^{n+1})^{ad} \to \overline{\mathcal{E}\ell\ell}(Np^n)^{ad} \to \ldots$. Although limits of adic spaces do not in general work nicely[2], one can make the following definition from [SW13] Definition 2.4.1:

Definition 2.2. If X_n is an inverse system of adic spaces, and X is an adic space, we say that $X \sim \varprojlim_n X_n$ if there is a system of maps $X \to X_n$ inducing a homeomorphism $|X| \simeq \varprojlim_n |X_n|$ on underlying spaces, and there is an affine open cover of X by $\text{Spa}(A, A^+)$ such that

$$\varinjlim_{(A_i, A_i^+) \to (A, A^+)} A_i \to A$$

has dense image, for $\text{Spa}(A_i, A_i^+)$ running over affinoid open subsets of the X_i.

Proposition 2.3 ([SW13], Proposition 2.4.5)**.** *If X_n is an inverse system of adic spaces with qcqs transition maps, and there is a perfectoid space X with $X \sim \varprojlim_n X_n$, then X is determined uniquely up to unique isomorphism by this property.*

Proof. Although in Proposition 2.4.5 of [SW13], this is only stated for X_n over a perfectoid field, we note that this is not actually used in the proof, and the proof goes through verbatim. □

With this in mind, we can define:

Definition 2.4. The **modular curve of infinite level** $X(Np^\infty)$, if it exists, is the perfectoid space with $X(Np^\infty) \sim \varprojlim_n \overline{\mathcal{E}\ell\ell}(Np^n)^{ad}_\eta$.

This defines the modular curve at infinite level over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, and is studied in detail (along with more general Shimura curves) in ([Sch15]). We will proceed to construct an "integral perfectoid" modular curve at infinite level, following Lurie [Lur20] (see also, [Cam21]), which pulls back to the one defined by Scholze on the generic fiber. This brings us to the main result of this section:

[1] Usually the procedure is done with a formal scheme, but nothing prevents us from lifting a formal stack to an adic stack. If $N \geq 5$ is coprime to p, then $\overline{\mathcal{E}\ell\ell}(N)$ is represented by an actual scheme, and this procedure yields an actual adic space.

[2] Which was one of the main motivations for Clausen-Scholze's definition of an analytic ring, but this is a digression.

Theorem 2.5. *Then there exists an "integral perfectoid" modular curve at infinite level, which we denote by* $\overline{\mathcal{E}\ell\ell}(Np^\infty)^{ad}$*, with the property that* $\overline{\mathcal{E}\ell\ell}(Np^\infty)^{ad} \sim \varprojlim_n \overline{\mathcal{E}\ell\ell}(Np^n)^{ad}$.

Proof. Lurie proceeds in a number of steps, and we will attempt to sketch the argument, referring to [Lur20] for the complete proof. To begin, we work with $\overline{\mathcal{E}\ell\ell}(Np^n)$ on the level of stacks over $\mathbb{Z}_p$, since all the relevant structure can already be defined here, and then consider the inverse limit, a DM stack $\overline{\mathcal{E}\ell\ell}(Np^\infty)$ over $\mathbb{Z}_p$. In order to show that an appropriate étale cover will become a cover by affinoid perfectoids upon passing to adic spaces, Lurie proves the following:

Lemma 2.6. *For* $n \geq 2$*, there is a unique map* Θ *making the following diagram commute*

$$\begin{array}{ccc} \overline{\mathcal{E}\ell\ell}(Np^n)_{p=0} & \xrightarrow{\ \varphi\ } & \overline{\mathcal{E}\ell\ell}(Np^n)_{\pi=0} \\ \downarrow & \overset{\Theta}{\swarrow} & \downarrow \\ \overline{\mathcal{E}\ell\ell}(Np^{n-1})_{p=0} & \xrightarrow{\ \varphi\ } & \overline{\mathcal{E}\ell\ell}(Np^{n-1})_{\pi=0} \end{array}$$

where φ *denotes the absolute Frobenius,* $\overline{\mathcal{E}\ell\ell}(Np^n)$ *is considered as fibered over* $\mathrm{Spec}(\mathbb{Z}_p[\zeta_{p^n}])$*, and* $\pi = (\zeta_{p^2} - 1)^{p-1}$ *is an element with* $\pi^p = up$ *for a unit* u*.*

The idea proceeds by using that the maps φ are flat, so one can define Θ via fppf descent. Thus, it suffices to replace the top row by $\mathcal{E}\ell\ell^{ord}_{p=0} \xrightarrow{\varphi} \mathcal{E}\ell\ell^{ord}_{\pi=0}$, since the image of

$$\mathcal{E}\ell\ell^{ord}_{p=0} \times_{\mathcal{E}\ell\ell^{ord}_{\pi=0}} \mathcal{E}\ell\ell^{ord}_{p=0} \to \overline{\mathcal{E}\ell\ell}(Np^n)_{p=0} \times_{\overline{\mathcal{E}\ell\ell}(Np^n)_{\pi=0}} \overline{\mathcal{E}\ell\ell}(Np^n)_{p=0}$$

is scheme-theoretically dense. Finally, using properties of ordinary elliptic curves one can construct the map explicitly (see Lurie section 2 [Lur20]).

In particular, φ induces an equivalent $\overline{\mathcal{E}\ell\ell}(Np^\infty)_{p=0} \to \overline{\mathcal{E}\ell\ell}(Np^\infty)_{\pi=0}$, which, interpreted on an open cover by Rs, means that there is some $\pi \in R$, $\pi^p | p$, and $\varphi : R/\pi \xrightarrow{\sim} R/\pi^p = R/p$. Passing to adic spaces, which amounts to p-completing these Rs to get an open cover, we get an open cover by "integral perfectoid" algebras $\mathrm{Spa}(R, R)$, with $\mathrm{Spa}(R[1/p], R)$ an actual affinoid perfectoid space, thus giving rise to $X(Np^\infty)$ on the generic fibers. □

This allows one to define p-adic modular forms at infinite level, studied more in depth in ([Heu20], [Cam21]). We will examine an analogue of Theorem 2.5 in the next section when we construct a deformation of Katz's rigid analytic modular curves.

§3. An Adic Deformation of Katz's Rigid Analytic Modular Curves

Let's recall that Katz's in [KM16] examined, for R_0 any p-adically complete ring, and any $r \in R_0$, the moduli problem $\overline{\mathcal{E}\ell\ell}(N,r)$, which gives rise to a formal stack which sends an R_0-algebra S on which p is nilpotent to the groupoid of triples $(E/S, \alpha_N, Y)$, where E/S is an elliptic curve, $\alpha_N : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$ is a full N level structure, and $Y \in \omega_S^{\otimes 1-p}$ is an element with $YE_{p-1} = r$. One can get a sort of universal case of this construction by taking R_0 to be the integral Tate algebra $\mathbb{Z}_p\langle T\rangle$, and $r = T$. This variant can certainly be considered in place of the following, which would give rise to a moduli stack one could call $\overline{\mathcal{E}\ell\ell}(N)\langle T\rangle$, fibered over $\mathrm{Spa}(\mathbb{Z}_p\langle T\rangle)$, whose rigid generic fiber is fibered over $\mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$, the adic closed unit disk, but we consider a variant where we impose that T be topologically nilpotent, which we feel is closer to overconvergent modular forms defined using powers p^r of p.

Definition 3.1. Let $\overline{\mathcal{E}\ell\ell}(N)[[T]]$ denote the formal moduli stack (or rather the DM compactification of such) which sends a $\mathbb{Z}_p[T]$-algebra S on which p and T are nilpotent to triples $(E/S, \alpha_N, Y)$, where E/S is an elliptic curve, α_N is a full N-level structure on E, and $Y \in \omega^{\otimes 1-p}$ is such that $YE_{p-1} = T$. We can take the associated adic space (or adic stack if $N < 5$) $\overline{\mathcal{E}\ell\ell}(N)[[T]]^{ad}$, which lives over $\mathrm{Spa}(\mathbb{Z}_p[[T]])$.

Remark 3.2. For $N \geq 5$, $\overline{\mathcal{E}\ell\ell}(N)[[T]]$ comes from an actual Scheme by results of Katz, but this time we are considering it as a sheaf on $\mathbb{Z}_p[T]$-algebras where both p and T are nilpotent in order to end up with a formal stack over $\mathrm{Spf}(\mathbb{Z}_p[[T]])$.

We recall from [SW20] the picture associated to $\mathrm{Spa}(\mathbb{Z}_p[[T]])$, there is a unique non-analytic point $x_{\mathbb{F}_p} : \mathrm{Spa}(\mathbb{F}_p) \to \mathrm{Spa}(\mathbb{Z}_p[[T]])$, and there is a surjective continuous map from the underlying space of $\mathcal{Y} := \mathrm{Spa}(\mathbb{Z}_p[[T]]) \backslash x_{\mathbb{F}_p}$ to $[0,\infty]$ taking a point x with maximal generalization $\tilde{x} : \mathbb{Z}_p[[T]] \to \mathbb{R}_{\geq 0}$, to

$$\kappa(x) := \frac{\log_p(|T(\tilde{x})|)}{\log_p(|p(\tilde{x})|)}.$$

So, we can naturally split up our moduli space into the fiber over the non-analytic point, which is a moduli stack over $\mathrm{Spa}(\mathbb{F}_p, \mathbb{F}_p)$, which recovers the (adic space associated to the) moduli problem sending an $\mathbb{F}_p$-algebra R to the groupoid of triples $(E/R, \alpha_N, Y)$, with E/R an elliptic curve, α_N a full level structure, and $Y \in \omega^{\otimes 1-p}$ such that $YE_{p-1} = 0$.[3] Finally, we define:

Definition 3.3. Let $X(N)[[T]]$ denote the fiber of the moduli space $\overline{\mathcal{E}\ell\ell}(N)[[T]]$ over $\mathcal{Y} \subseteq \mathrm{Spa}(\mathbb{Z}_p[[T]])$. It (in the cases it is an actual adic space) is an analytic adic space over $\mathcal{Y}$, and in particular, we have a map $|X(N)[[T]]| \to [0,\infty]$.

We define a variant on this, which seems to be closer to overconvergent modular forms:

Definition 3.4. Let $X^{\dagger}(N)[[T]]$ denote the fiber of $X(N)[[T]]$ over $(0,\infty]$, i.e., the open subscheme arising as the fiber over $\mathrm{Spa}(\mathbb{Q}_p[[T]], \mathbb{Z}_p[[T]])$. For $r \in \mathbb{Z}_{>0}$, let $X_r^{\dagger}(N)[[T]]$ denote the fiber of $X(N)[[T]]$ over $1/r \in [0,\infty]$. This arises as the fiber over the open subscheme which is the intersection of rational subsets of Y: $\{x : |T(x)| \leq |p^r(x)| \neq 0\} \cap \{x : |p^r(x)| \leq |T(x)| \neq 0\}$.

[3] In particular, if E/R is ordinary, then $Y = 0$ necessarily. However, at supersingular points and cusps, this adic stack is not as easy to describe.

Our notation of $X_r^\dagger(N)[[T]]$ is perhaps a bit misleading, since this is not the curve X_r^{rig} defined in class, but an adic thickening of it. The reason for this is that the fiber over r of the adic space $\mathcal{Y}$ can specialize T to any point with $|T(x)| = |p^r(x)| \neq 0$, but of course T and p^r differ only by a unit in this case, so that $X_r^\dagger(N)[[T]]$ is a "$\mathbb{G}_m^{ad}$" thickening of Katz's moduli space. In particular, specializing to the point $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, $T \mapsto p^r$ gives us the rigid space X_r^{rig} discussed in class.

There is another option for the r-overconvergent modular curve, which is a bit bigger than the one defined above, but is perhaps more in the spirit of the definition of X_r^{rig}. Recall from [SW20] section 4 that $\mathcal{Y}$ has rational subsets, for a/b a positive rational number (a,b>0 integers), $\mathcal{Y}_{[a/b,\infty]} = \mathrm{Spa}(\mathbb{Q}_p\langle T, T^a/p^b\rangle)$. We can define, for $r \in (0,\infty]$

Definition 3.5. The overconvergent adic modular curve of growth r, $X^{\dagger,ad}(N)[[T]]$ is the fiber of $X^\dagger(N)[[T]]$ over $[r,\infty]$.

We note that the fiber over the point $0 \in [0,\infty]$ gives us a moduli stack over $\mathrm{Spa}(\mathbb{F}_p((T)), \mathbb{F}_p[[T]])$, which we may think of as sending an affinoid $\mathbb{F}_p$-algebra (A, A^+) together with a choice of topologically nilpotent unit T to a triple $(E/A^+, \alpha_N, Y)$ as before (E/A^+ isn't really an elliptic curve moreso as some inductive system of projective systems of elliptic curves over $A_\circ/T^n$ for open bounded subrings of A^+). In particular, since T is a unit, all the elliptic curves we get here have to be ordinary, so that the fiber over 0 almost recovers some version of X^{ord}, although it lands in characteristic p.

Definition 3.6. Define the space of **weight *k* overconvergent adic modular forms of level *N*** to be the $\mathbb{Q}_p$-vector space defined by $\mathcal{M}_k^{\dagger,ad}(N) := H^0(X^\dagger(N)[[T]], \omega^{\otimes k})$.

The choice to exclude 0 when defining $X^\dagger(N)[[T]]$ was to avoid the "ordinary forms" from contributing, though there are several variants on this definition and it's unclear which would be the correct one, though if we did the analogous construction for the full curve $X(N)[[T]]$, then we would no longer have a $\mathbb{Q}_p$-vector space. By our previous discussion, there is a map $\mathcal{M}_k^{\dagger,ad}(N) \to \mathcal{M}_k^\dagger(N,r)$ for every $r \in \mathbb{Z}$, where the target is defined in the sense of Katz. Since we can forget $X(N)[[T]] \to \overline{\mathcal{E}\ell\ell}(N)^{ad}$, which extends to $X^\dagger(N)[[T]] \to \overline{\mathcal{E}\ell\ell}(N)^{ad}_\eta$, we get that the map from $\mathbb{Q}_p$-modular forms (those arising from the full X^{rig} in Katz's definitions) $\mathcal{M}_k(N) \to \mathcal{M}_k^\dagger(N,r)$ to r-overconvergent modular forms factors over $\mathcal{M}_k^{\dagger,ad}(N)$.

Now, we'll explain a variant of modular curves at infinite level in these constructions, following Lurie's proof making necessary modifications to construct an integral perfectoid space $X(Np^\infty)[[T^{1/p^\infty}]]$ fibered over $\mathrm{Spa}(\mathbb{Z}_p^{cyc}[[T^{1/p^\infty}]], \mathbb{Z}_p^{cyc}[[T^{1/p^\infty}]])$. To begin, let's try and naively adapt Lurie's approach on the level of formal stacks:

Theorem 3.7 (Wrong Theorem)**.** *For $n \geq 2$, there is a unique map Θ making the following diagram of formal moduli stacks commute*

$$\begin{array}{ccc} \overline{\mathcal{E}\ell\ell}(Np^n)[[T]]_{p=0} & \xrightarrow{\varphi} & \overline{\mathcal{E}\ell\ell}(Np^n)[[T]]_{\pi=0} \\ \downarrow & \overset{\Theta}{\swarrow} & \downarrow \\ \overline{\mathcal{E}\ell\ell}(Np^{n-1})[[T]]_{p=0} & \xrightarrow{\varphi} & \overline{\mathcal{E}\ell\ell}(Np^{n-1})[[T]]_{\pi=0} \end{array}$$

The reason why this is wrong is that when we have our elliptic curve E/S over our $\mathbb{F}_p$-algebra S with $Y \in \omega^{\otimes 1-p}$ $YE_{p-1} = T$, then passing to Frobenius, φ, our new triple $(\varphi(E/S), \varphi(\alpha_{Np^n}), \varphi(Y))$ has $\varphi(Y)E_{p-1} = T^p$. Since T is a nilpotent element, there cannot be a lift Θ making this commute, as there may be many choices of T in S giving rise to the same T^p. The issue here comes from the T variable, of course, the adic space associated to $\mathrm{Spf}(\mathbb{Z}_p[[T]])$ is $\mathrm{Spa}(\mathbb{Z}_p[[T]])$, which is very far from perfectoid. Instead, what we need to do is use a diagram of the following form:

$$\begin{array}{ccc} \overline{\mathcal{E}\ell\ell}(Np^n)[[T^{1/p^n}]]_{p=0} & \xrightarrow{\varphi} & \overline{\mathcal{E}\ell\ell}(Np^n)[[T^{1/p^{n-1}}]]_{\pi=0} \\ \downarrow & \swarrow_{\Theta} & \downarrow \\ \overline{\mathcal{E}\ell\ell}(Np^{n-1})[[T^{1/p^{n-1}}]]_{p=0} & \xrightarrow{\varphi} & \overline{\mathcal{E}\ell\ell}(Np^{n-1})[[T^{1/p^{n-2}}]]_{\pi=0}. \end{array}$$

The vertical maps here are going to be forced upon us by the construction. We can now begin with the actual construction:

Theorem 3.8. *Let N be a natural number. Then, for $n \geq 2$, we have a natural commutative diagram of moduli stacks lifting Lurie's diagram*

$$\begin{array}{ccc} \overline{\mathcal{E}\ell\ell}(Np^n)[[T^{1/p^n}]]_{p=0} & \xrightarrow{\varphi} & \overline{\mathcal{E}\ell\ell}(Np^n)[[T^{1/p^{n-1}}]]_{\pi=0} \\ \downarrow & \swarrow_{\Theta} & \downarrow \\ \overline{\mathcal{E}\ell\ell}(Np^{n-1})[[T^{1/p^{n-1}}]]_{p=0} & \xrightarrow{\varphi} & \overline{\mathcal{E}\ell\ell}(Np^{n-1})[[T^{1/p^{n-2}}]]_{\pi=0}. \end{array}$$

where the vertical maps are induced from $\overline{\mathcal{E}\ell\ell}(Np^n)[[T^{1/p^n}]] \to \overline{\mathcal{E}\ell\ell}(Np^{n-1})[[T^{1/p^{n-1}}]]$, $(E/S, \alpha_{Np^n}, Y) \mapsto (E/S, \alpha_{Np^{n-1}}, YT^{(p-1)/p^n})$, and Θ is uniquely determined.

sketch of proof. The horizontal maps in this case are still flat[4], so we may restrict to the schematically dense substack $\mathcal{E}\ell\ell^{ord}[[T^{1/p^n}]]_{p=0}$ as in [Lur20], to construct our map. The top horizontal map takes $(E/S, \alpha, Y)$ to $(E^{(p)}, F\alpha, Y^{(p)})$, and by Lurie's construction, there is a map Θ defined on pairs $(E/S, \alpha)$, taking E to a curve E' together with an étale isogeny of degree p $E \to E'$. By [Lur20], this induces an isomorphism $E \simeq (E')^{(p)}$, and then we may pull back Y along the relative Frobenius $F : E' \to (E')^{(p)}$ to define $F^*Y \in \omega_{E'}^{\otimes 1-p}$ with $F^*YE_{p-1} = T^{1/p^{n-1}}$. □

Corollary 3.9. *There is an integral perfectoid modular curve at infinite level $X(Np^\infty)[[T^{1/p^\infty}]]$ with growth T, which is fibered over the integral affinoid perfectoid space* $\mathrm{Spa}(\mathbb{Z}_p^{cyc}[[T^{1/p^\infty}]], \mathbb{Z}_p^{cyc}[[T^{1/p^\infty}]])$.

In particular, we can define, analogous to Definition 3.4:

Definition 3.10. Let $X^\dagger(Np^\infty)[[T^{1/p^\infty}]]$ denote the fiber of $X(Np^\infty)[[T^{1/p^\infty}]]$ over $\{x : |p(x)| \neq 0\}$, which is a perfectoid space fibered over $\mathrm{Spa}(\mathbb{Q}_p^{cyc}[[T^{1/p^\infty}]], \mathbb{Z}_p^{cyc}[[T^{1/p^\infty}]])$.

Analogously to our discussion before, this allows us to now define overconvergent modular forms at infinite level. The actual exploration of these concepts is outside the scope of this paper (and as with everything in this section, I do not know if it has been explored or not).

[4]The fact that the power of T differs is crucial for this fact to hold.

§4. Some Questions

Here we present some questions on the above that may be interesting for further exploration:

Question 4.1. Are the maps $\mathcal{M}_k^{\dagger,ad}(N) \to \mathcal{M}_k^{\dagger}(N,r)$ surjective for $r \in \mathbb{Z}_{>0}$? What does the ring $\mathcal{M}^{\dagger,ad}(N)$ look like?

Question 4.2. If we do a variant on the construction of $X^{\dagger}(N)[[T]]$ by basechanging to define $X^{\dagger}(N)[[T^{1/p^{\infty}}]]$, then the isomorphism induced by $T \mapsto T^p$ (on the affine level) gives us a compatible system of diagrams (here we must consider our rigid spaces as living over $\mathrm{Spa}(\mathbb{Q}_p[p^{1/p^{\infty}}], \mathbb{Z}_p[p^{1/p^{\infty}}])$ via basechange to make sense of the vertical maps):

$$\begin{array}{ccc} X_r^{rig}(N) & \longrightarrow & X_{pr}^{rig} \\ \downarrow & & \downarrow \\ X^{\dagger}(N)[[T^{1/p^{\infty}}]] & \xrightarrow{\sim} & X^{\dagger}(N)[[T^{1/p^{\infty}}]] \end{array}$$

Thus, we can define modular forms $\mathcal{M}^{\dagger,ad,\infty}(N) := H^k(X^{\dagger}(N)[[T^{1/p^{\infty}}]], \omega^{\otimes k})$, and we get a system of maps $\mathcal{M}^{\dagger,ad,\infty}(N) \to \mathcal{M}_k^{\dagger}(N,r)$ compatible with the transition maps between these. In particular, the image of $\mathcal{M}^{\dagger,ad,\infty}(N)$ should be contained in all of our $\mathcal{M}_k^{\dagger}(N,r)$, so should be rather small. Can one use this to say something about $\mathcal{M}_k^{\dagger,ad}(N)$ as we have defined it?

Question 4.3. How does Hecke equivariance factor in to the above story? Does the adic framework provide a convenient location for which one can study modular forms via the above techniques?

Question 4.4. How does the story of §3 adapt to the case of Shimura curves similarly to what was studied by Scholze in [Sch15]?

Question 4.5. From Definition 3.10, we can define overconvergent modular forms at infinite level. How do these relate to other notions of modular curves at infinite level? How well does the classical theory of overconvergent modular forms adapt to this context?

Although this final project takes a very different form to final projects I have submitted in the past, I hope that it is satisfactory. I would have liked to prove more instead of leaving an entire section of questions I have, but due to a lack of time, working out some formalism seen in §3 is all we will be able to do for now. Although it was mostly definitions, we do hope the topic is at least mildly interesting to the reader.

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