A DEDEKIND APPROACH TO EISENSTEIN COCYLES IN MOTIVIC COHOMOLOGY

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ABSTRACT. In this note, we will discuss an approach to certain cocyles constructed by Sharifi-Venkatesh [SV23] through examining the motivic cohomology of Dedekind schemes. In the first section, we review Gysin filtrations on motivic cohomology (which works for any algebraic cohomology theory admitting a Thom isomorphism). With this background in play, we specialize to the case of schemes over the integers $\mathbb{Z}$, and describe the procedure to recover the cocycles constructed in [SV23]. We end by expanding on the construction of this class through "naive equivariant motivic cohomology," noting that there are $\mathrm{GL}_2(\mathbb{Z})$-equivariant actions everywhere, which gives rise to the class of the cocycle Θ in a natural way.

CONTENTS

§1. REVIEW OF GYSIN FILTRATIONS

Fix a commutative ring k of finite Krull dimension. We will work with the motivic homotopy category Sp_k^{mot}[1], although since we only care about motivic cohomology, the reader may freely assume the constructions are taking place in the category DM_k of motives over k. With this in mind, we will often abusively write a scheme X for its suspension spectrum $\Sigma^\infty_+ X$ (respectively motive $M(X)$), at least when no confusion is likely to arise. Given a stable ∞-category $\mathcal{C}$, a filtered object in $\mathcal{C}$ will mean a functor $\mathbb{Z}^{\leq} \to \mathcal{C}$, which we represent as a diagram

$$C_\bullet = \ldots \to C_{i+1} \to C_i \to C_{i-1} \to \ldots.$$

Any filtered object gives rise to an associated graded obtained by taking $C_i^{gr} := \mathrm{cofib}(C_{i+1} \to C_i)$, whose homotopy groups (supposing $\mathcal{C}$ has a good notion of homotopy groups) serve as the E_1-page of a spectral sequence. Under suitable conditions on $C_\bullet$ (namely, that $\varprojlim C_\bullet \simeq 0$), this spectral sequence will converge to the homotopy groups of $\varinjlim C_\bullet$. We refer to [BHS22, §Appendix B] for a more in depth discussion of the formalism of filtered objects.

Construction. Consider a smooth equi-dimensional k-scheme X, and closed subschemes $X = Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_n \supseteq Z_{n+1} = \emptyset$, such that Z_i has pure codimension i, and $Z_i \backslash Z_{i+1}$ is smooth for all i. To such data, encoded by $\{Z_i\}$, we associate a filtered object $\{Z_i\}^{gys}_\bullet \in \mathrm{Sp}_k^{mot}$, which we will term the *Gysin filtration* for $\{Z_i\}$, defined by:

$$\ldots \xrightarrow{id} X \xrightarrow{id} X \to \mathrm{cofib}(X\backslash Z_1 \to X) \to \ldots \to \mathrm{cofib}(X\backslash Z_n \to X) \to 0 \to \ldots, \tag{1.1}$$

where the final X which appears in the diagram lives in grading 0 (that is $\{Z_i\}^{gys}_0 = X$).

[1] This can be defined as Nisnevich sheaves of finite type smooth k-schemes valued in the (∞-)category of spectra which are $\mathbb{A}^1$-invariant- that is, the natural map $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1 \times X)$ is an equivalence for all smooth k-schemes X.

Given a Gysin filtration on X, and a motivic spectrum $\mathcal{Y}$, we can consider the object $\mathrm{Hom}(\{Z_i\}_\bullet^{gys}, \mathcal{Y})$, defined by taking morphisms to Y pointwise in the diagram (1.1). The resulting object is a filtration on the Y-based cohomology of X. This leads to:

Definition 1.1. Given a motivic spectrum $\mathcal{Y}$ and a Gysin filtration on X, the *$\mathcal{Y}$-based Gysin spectral sequence* for X is the spectral sequence associated to the filtered object $\mathrm{Hom}(\{Z_i\}_\bullet^{gys}, \mathcal{Y})$.

To describe the E_1-page and relate this back to our usual Gysin spectral sequence, we use the homotopy purity theorem:

Theorem 1.2 (Homotopy Purity, [MV99] Theorem 3.2.23)**.** *Let k be a Noetherian ring of finite Krull dimension. If $Z \to X$ is closed embedding of smooth k-schemes, then there is an equivalence* $\mathrm{cofib}(X\backslash Z \to X) \simeq \mathrm{Th}(N_Z(X))$. *Here $N_Z(X) \to Z$ is the normal bundle of Z in X, and* $\mathrm{Th}(E)$ *is the Thom space of a vector bundle E, which may be defined as the homotopy quotient of E by the complement of the zero section, $E/E - i_0(X)$.*

Corollary 1.3. *There is an equivalence*

$$\mathrm{fib}(\mathrm{cofib}(X\backslash Z_i \to X) \to \mathrm{cofib}(X\backslash Z_{i+1} \to X)) \simeq \mathrm{Th}_{Z_i\backslash Z_{i+1}}(X\backslash Z_{i+1}).$$

Proof. Since $Z_i\backslash Z_{i+1}$ is smooth and closed in $X\backslash Z_{i+1}$, the homotopy purity theorem applies to the canonical inclusion $X\backslash Z_i \to X\backslash Z_{i+1}$ to tell us the cofiber is equivalent to $\mathrm{Th}_{Z_i\backslash Z_{i+1}}(X\backslash Z_{i+1})$. A diagram chase (namely the octahedral axiom) applied to $X\backslash Z_i \to X\backslash Z_{i+1} \to X$ proves the claim. □

In particular, if the normal bundle $N_{Z_i\backslash Z_{i+1}}(X\backslash Z_{i+1})$ is trivial, then the Thom space is equivalent to $(\mathbb{P}^1)^{\wedge i} \wedge (Z_i\backslash Z_{i+1})$, which is a shift of $Z_i\backslash Z_{i+1}$ by $S^{2i,i}$, and

$$H^{p,q}(\mathrm{Th}(N_{Z_i\backslash Z_{i+1}}(X\backslash Z_{i+1})), \mathcal{Y}) \simeq H^{p-2i,q-i}(Z_i\backslash Z_{i+1}, \mathcal{Y}),$$

for any motivic spectrum $\mathcal{Y}$. In general though, there is no reason to expect the Gysin filtration to be easy to define unless our cohomology theory supports a Thom isomorphism (which occurs e.g., if it is oriented). However, in these cases, we recover:

Corollary 1.4. *If $\mathcal{Y}$ is an oriented motivic spectrum, then the E_1-page of a $\mathcal{Y}$-based Gysin spectral sequence for X has the form*

$$E_1^{p,q} = H^{q-p,r-p}(Z_p\backslash Z_{p+1}, \mathcal{Y}) \implies H^{p+q,r}(X, \mathcal{Y}).$$

□

These filtered objects are contravariantly functorial under refining the filtration, so by passing to an inverse limit of filtered objects one can recover a Coniveau filtration (when we dualize to cohomology, the inverse limit defining this filtered object will become a filtered colimit, and since filtered colimits commute with taking homotopy groups, this will recover the Coniveau spectral sequence in the sense of [SV23]). In the sequel, we will only apply this in the case that $\mathcal{Y}$ represents ordinary motivic cohomology, where Corollary 1.4 applies, not requiring the full generality of the above construction.

§2. The Cocycle Θ

We now focus on specializing the generalities in §1 to our case of interest. For the remainder of this section, we work with schemes over $\mathbb{Z}$ unless otherwise stated.

Construction [SV23] For a primitive $(a, c) \in \mathbb{Z}^2$, let $S_{a,c} = \ker(\mathbb{G}_m^2 \xrightarrow{(z_1,z_2)\mapsto z_1^a z_2^c} \mathbb{G}_m)$ be the subgroup scheme defined by this kernel. For each index set $I \subseteq \mathbb{Z}^2$ of primitive elements not containing more than one representative of any element of $\mathbb{Q}$, we define

$$S_I := \cup_{(a,c)\in I} S_{a,c},$$

and

$$T_I := \cup_{(a,c)\neq(b,d)\in I} S_{a,c} \cap S_{b,d}.$$

These are closed subschemes of $\mathbb{G}_m^2$, which give rise to a Gysin filtration $\mathbb{G}_m^2 \supseteq S_I \supseteq T_I$, whose associated filtered object we will call Z_I.

If two index sets I and J have the same image in $\mathbb{P}^1(\mathbb{Q})$, then we have natural isomorphisms $S_I \simeq S_J$ and $T_I \simeq T_J$, yielding a natural equivalence $Z_I \simeq Z_J$. Furthermore, if $I \supseteq J$, there are natural maps $X\backslash S_J \to X\backslash S_I$, $X\backslash T_J \to X\backslash T_I$, and thus $Z_J \to Z_I$. We can therefore take the inverse limit across all possible index sets I to get a filtered object $Z_\bullet := \varprojlim_I Z_I$. There is a natural action of $\mathrm{GL}_2(\mathbb{Z}) =: \Gamma$ on the indexing set I, which descends to a Γ-action on $\varprojlim_I \mathbb{G}_m^2\backslash S_I$, $\varprojlim_I \mathbb{G}_m^2\backslash T_I$, and to $Z_\bullet$. This action will be examined more closely next section where we will use "naive equivariant motivic cohomology" to construct Θ in a very natural manner, but for now, we will only use that it provides a Γ-action on the homotopy groups.

We wish to compare $Z_\bullet$ with the constructions from [SV23]. For this, we will use the following computational lemmas:

Lemma 2.1. *The cohomology group $H^{3,2}(Z_1)$ agrees with what [SV23] term $K_{2/\mathbb{Z}}/H^{2,2}(\mathbb{G}_m^2)$, where $K_{2/\mathbb{Z}} := \varinjlim_I H^{2,2}(\mathbb{G}_m^2\backslash S_I)$.*

Proof. First, note that there is a fiber sequence

$$\varprojlim_I \mathbb{G}_m^2\backslash S_I \to \mathbb{G}_m^2 \to Z_1,$$

which in turn provides us with a long exact sequence:

$$\ldots \to H^{2,2}(\mathbb{G}_m^2) \to \varinjlim H^{2,2}(\mathbb{G}_m^2\backslash S_I) = K_{2/\mathbb{Z}} \to H^{3,2}(Z_1) \to H^{3,2}(\mathbb{G}_m^2) \to \ldots.$$

Since the motivic cohomology of $\mathbb{Z}$ vanishes in degrees $H^{3,2}$, $H^{2,1}$ and $H^{1,0}$, we find that $H^{3,2}(\mathbb{G}_m^2) \simeq H^{3,2}(\mathbb{Z}) \oplus H^{2,1}(\mathbb{Z})^{\oplus 2} \oplus H^{1,0}(\mathbb{Z}) = 0$, and the claim follows. □

Lemma 2.2. *The cohomology $H^{*,2}(Z_2)$ vanishes unless $* = 4$, in which case $H^{4,2}(Z_2) \simeq \varinjlim_I H^{0,0}(T_I) =: K_{0/\mathbb{Z}}$.*

Proof. From the homotopy purity theorem, using smoothness of T_I, we find that $Z_2 \simeq \varprojlim_I \mathrm{Th}(T_I)$, and the claim then follows from the Thom isomorphism and the fact that $H^{*,0}$ of smooth schemes is concentrated in degree 0. □

With these two lemmas in place, we are ready to get the main short exact sequence (which could be obtained from the spectral sequence itself as well) of interest.

Theorem 2.3. *By taking cohomology of the fiber sequence* $\varprojlim_I \mathrm{Th}_{S_I\backslash T_I}(\mathbb{G}_m^2\backslash T_I) \to Z_1 \to Z_2$*, one obtains an exact sequence*

$$0 \to K_{2/\mathbb{Z}}/H^{2,2}(\mathbb{G}_m^2) \to \varinjlim_I H^{1,1}(S_I\backslash T_I) =: K_{1/\mathbb{Z}} \to K_{0/\mathbb{Z}}.$$

Proof. Applying $H^{*,2}$ to the fiber sequence in the statement provides a long exact sequence

$$\ldots \to H^{3,2}(Z_2) \to H^{3,2}(Z_1) \to \varinjlim_I H^{1,1}(S_I\backslash T_I) \to H^{4,2}(Z_2) \to H^{4,2}(Z_1) \to \ldots.$$

By Lemmas 2.1 and 2.2, we are done. □

Remark 2.4. Right exactness is harder to obtain at an integral level, but may be provable by examining Picard groups of the unions of $S_{a,c}$. The cokernel of $K_{1/\mathbb{Z}} \to K_{0/\mathbb{Z}}$ will be precisely the entry $E_2^{2,2}$ on the $(2,2)$ slot of the Coniveau-type spectral sequence associated to our filtration. This will contribute to $H^{4,2}(\mathbb{G}_m^2) = 0$, so must vanish somewhere in the spectral sequence. Since the grading on the d_r-differential with gradings as in [SV23] is $(r, 1-r)$, the only possible contribution is if a class in grading 0 hits this group on the E_2-page. This class would have to live in $E_2^{0,3}$, which is $\varinjlim_I H^{3,2}(\mathbb{G}_m^2\backslash S_I)$, though this isomorphism could also be obtained without recourse to the spectral sequence.

Although we do not have right exactness entirely from the above, we can still note that the class $1 \in \mathbb{G}_m$ is in the image of this last map, just by an explicit construction with symbols, so pulling back along the $\mathrm{GL}_2(\mathbb{Z})$-equivariant inclusion of this fixed point yields the extension class Θ from [SV23]. Proposition 3.3.1 of [SV23] goes through practically verbatim to our situation.

Trace maps can be defined in our situations on the level of motivic complexes, using the orientation on the spectrum representing integral motivic cohomology to define proper push-forwards ([Spi13], [MV99], [Lev08]). The results of SV §4.1 go through almost verbatim in this situation, though we the details for time constraints. The only remark we make is that for considering $M^{(0)}$, one should define this as the subgroup of M fixed by all trace operators $[p]_*$ for all primes p. This could have been done in [SV23] as well without changing the results needed to lift the cocycle to $K_2/\{-z_1, -z_2\}$ (even though there they avoid the case p is equal to the characteristic).

Construction. For the specializations to $\mathrm{Spec}(\mathbb{Z}[1/N, \mu_N])$-points, one can proceed by the above constructions over the ring $\mathbb{Z}[1/N]$, where our indexing sets I are forced to avoid the point $\mathrm{Spec}(\mathbb{Z}[1/N, \mu_N]) \xrightarrow{(1,\zeta_N)} \mathbb{G}^2_{m,/\mathbb{Z}[1/N]}$.

The reason we need to invert N is that, otherwise, for some prime p dividing N, there is a point $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z}[\mu_N]) \to \mathbb{G}_m^2$ which will be in the intersection of every S_I with $\mathrm{Spec}(\mathbb{Z}[\mu_N])$. By inverting N, we allow an interesting theory to arise.

All in all, we can define $K_{2/\mathbb{Z}}(N) := \varinjlim_I H^{2,2}(\mathbb{G}^2_{m,/\mathbb{Z}[1/N]}\backslash S_I)$, with out limit taken over indexing sets I such that $S_I \cap \mathrm{Spec}(\mathbb{Z}[1/N, \mu_N]) = \emptyset$. This allows us to define a map

$$\mathrm{Spec}(\mathbb{Z}[1/N, \mu_N]) \to \varprojlim_I \mathbb{G}_m^2\backslash S_I \to \mathbb{G}_m^2,$$

providing a map $K_{2/\mathbb{Z}}(N) \to H^{2,2}(\mathbb{Z}[1/N, \mu_N])$. This map is equivariant for the natural $\widetilde{\Gamma}_1(N) := \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}) : N|c, d \equiv 1 \mod N\}$-action on the source and trivial action on the target. It also maps the (image of the) class $\{-z_1, -z_2\}$ to the class $\{-1, -\zeta_N\}$, using notation for these cup products as in [SV23].

The Galois action on $\mathrm{Spec}(\mathbb{Z}[1/N, \mu_N])$ intertwines the action of $\widetilde{\Gamma}_0 := \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}) : N|c\}$ on $\varinjlim_I \mathbb{G}_m^2 \backslash S_I$. Using the explicit cocycle $\Theta : \widetilde{\Gamma}_0 \to K_{2/\mathbb{Z}}(N)/\{-z_1, -z_2\}$ that can be constructed for similar reasons to (loccit), this map just constructed provides a cocycle $\Theta_N : \widetilde{\Gamma}_0(N) \to K_2(\mathbb{Z}[1/N, \mu_N])/(\widetilde{\Gamma}_0(N) \cdot \{-1, -\zeta_N\})$, as desired.

§3. The Equivariant Perpsective

In this section, we provide a slightly different route to the Sharifi-Venkatesh construction of a lift of the cocycle Θ to $K_2(k(\mathbb{G}_m^2)) \otimes \mathbb{Z}[1/6]$ over a field. For the purposes of this secction, we work over the base field $\mathbb{Q}$, although the construction should also work for finite fields with appropriate modifications.

We can construct a filtered motive (or motivic spectrum) $Z_\bullet$ as in §2, working over $\mathbb{Q}$ now, which comes with an action of $\Gamma := \mathrm{GL}_2(\mathbb{Z})$. Applying the functor $\mathrm{Hom}(-, H\mathbb{Z} \otimes S^{0,2})$, with $H\mathbb{Z}$ the motivic spectrum representing ordinary motivic cohomology (resp. homs into $\mathbb{Z}(2)$ if one prefers to work in the category of motives), we obtain a filtered object of the derived category $\mathcal{D}(\mathbb{Z})$

$$Y_\bullet = \ldots \to 0 \to Y_2 \to Y_1 \to Y_0 \xrightarrow{\sim} Y_{-1} \xrightarrow{\sim} \ldots,$$

equipped with a Γ-action, and such that the cohomology groups $H^*(Y_i) = H^{*,2}_{mot}(Z_i)$ compute motivic cohomology in weight 2. The associated spectral sequence to $Y_\bullet$ is precisely the Coniveau spectral sequence.

From here on out, we implicitly invert 6, writing $\mathbb{Z}' := \mathbb{Z}[1/6]$, working in $\mathcal{D}(\mathbb{Z}[1/6])$, which will make the analysis much easier, motivic cohomology will implicitly be taken with $\mathbb{Z}[1/6]$ coefficients unless otherwise stated. Let's describe some of these cohomology groups of $Y_\bullet$ explicitly:

Lemma 3.1. *$H^*(Y_0)$ is concentrated in degree 2, given by $H^{2,2}(\mathbb{Q}) \oplus H^{1,1}(\mathbb{Q})^{\oplus 2} \oplus H^{0,0}(\mathbb{Q})$.*

Proof. Since $H^{*,2}(\mathbb{G}_m^2) \simeq H^{*,2}(\mathbb{Q}) \oplus H^{*-1,1}(\mathbb{Q})^{\oplus 2} \oplus H^{*-2,0}(\mathbb{Q})$, we immediately get the second part. For the first, the only possible nonzero contributions are from $H^{0,2}(\mathbb{Q})$ and $H^{1,2}(\mathbb{Q})$. The spectral sequence from motivic cohomology to algebraic K-theory degenerates rationally, $K_3(\mathbb{Q})_\mathbb{Q} = 0$, and $K_2(\mathbb{Q}) \simeq K_2 \simeq H^{2,2}(\mathbb{Q})$, so both of these groups must have trivial rationalization. We can therefore check if they are zero one prime at a time. By the Beilinson-Lichtenbaum conjecture, this reduces us to computing étale cohomology. Any class in $H^{1,2}(\mathbb{Q})$ which is p-torsion will provide a class in $H^{0,2}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z})$, and since $H^{-1,2}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) = 0$, we need only show $H^{0,2}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) = 0$. But $H^{0,2}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) = H^0_{\acute{e}t}(\mathbb{Q}, \mu_p^{\otimes 2})$, and for $p > 3$, $\mu_p^{\otimes 2}$ has no Galois fixed points, so this group is zero, and we win. □

Since $H^*(Y_2)$ computes $H^{*-4,0}(\varprojlim T_I)$, we find that $H^*(Y_2)$ is concentrated in degree 4, where it is given by $\bigoplus_{x \in \mathbb{G}_m^2} \mathbb{Z}$. We have exact triangles:

$$Y_1 \to Y_0 \to H^{*,2}(k(\mathbb{G}_m^2)),$$

and

$$Y_2 \to Y_1 \to \bigoplus_D H^{*,1}(k(D))[-2].$$

Since $k(\mathbb{G}_m^2)$ is a field, its weight 2 motivic cohomology is concentrated in cohomological degrees $[0,2]$. The rings $k(D)$ are also fields, so the cohomology of the objects $H^{*,1}(k(D))[-2]$ are nonzero only in cohomological degree 3, where it is $H^{1,1}(k(D))$.

Proposition 3.2. *The object Y_1 has cohomology concentrated in degree 3, and the object $H^{*,2}(k(\mathbb{G}_m^2))$ has cohomology concentrated in degree 2.*

Proof. Using the second fiber sequence considered above, the fact that Y_2 has cohomology in degree 4, and $H^{*,1}(k(D))[-2]$ has cohomology concentrated in degree 3, we find that

Y_1 has cohomology concentrated in degrees $[3,4]$. Since Y_0 has cohomology only in degree 2, and the cohomology of $H^{*,2}(k(\mathbb{G}_m^2))$ is only possibly nonzero in degrees $[0,2]$, the first fiber sequence shows that Y_1 has cohomology concentrated in degree $[1,3]$. Thus, Y_1 has cohomology concentrated in degree 3, and then the first fiber sequence shows $H^{*,2}(k(\mathbb{G}_m^2))$ must have vanishing cohomology away from degree 2. $\square$

It is time to introduce the second complex, arising from the comparison [SV23] make to $\mathbb{G}_m^2\backslash\{1\}$. We can make the same sort of definitions as before, define $W_\bullet$ using the Coniveau filtration on $\mathbb{G}_m^2\backslash\{1\}$, this is a Γ-equivariant filtered motivic complex (or spectrum), which comes equipped with a Γ-equivariant map to $Z_\bullet$. Applying the functor $\mathrm{Hom}(-, H\mathbb{Z}\otimes S^{0,2})$ (or $\mathrm{Hom}(-,\mathbb{Z}(2))$) again, we obtain a filtered object in $\mathcal{D}(\mathbb{Z}[1/6])$ which we term $X_\bullet$, equipped with an action of Γ, and a Γ-equivariant map $Y_\bullet \to X_\bullet$. For the remainder of the paper, work in the category $\mathcal{D}(\mathbb{Z}[1/6])^{B\Gamma}$ of Γ-equivariant objects in $\mathcal{D}(\mathbb{Z}[1/6])$[2]

Lemma 3.3. *There is a natural fiber sequence $Y_i \to X_i \to \mathbb{Z}'[-3]$, where $\mathbb{Z}'$ carries the trivial Γ action, for $i \leq 2$.*

Proof. Since the cofibers of $Y_i \to Y_{i-1}$ and $X_i \to X_{i-1}$ agree for $i \leq 2$ (both giving $\bigoplus_D H^{*,1}(k(D))[-2]$, D a sum over divisors on $\mathbb{G}_m^2$ resp $\mathbb{G}_m^2\backslash\{1\}$, $H^{*,2}(k(\mathbb{G}_m^2))$, or 0), it suffices to prove the claim with $i = 0$. We use the homotopy purity theorem to get a cofiber sequence of motivic spectra $\mathbb{G}_m^2\backslash\{1\} \to \mathbb{G}_m^2 \to \mathrm{Th}_{\{1\}}(\mathbb{G}_m^2)$, note that $\mathrm{Th}_{\{1\}}(\mathbb{G}_m^2) \simeq \mathbb{P}^1 \otimes \mathbb{P}^2$, and then use that $H^{*,2}(\mathbb{P}^1 \otimes \mathbb{P}^2) = 0$ unless $* = 4$, in which case it is $\mathbb{Z}'$. $\square$

Now, we have a map $X_0 \to H^{*,2}(k(\mathbb{G}_m^2))$, and if we could split $X_0 \to \mathbb{Z}'[-3]$, this would allow us to define a map $\mathbb{Z}'[-3] \to H^{*,2}(k(\mathbb{G}_m^2))$, which would serve as a good candidate for $[\Theta] \in H^1(\Gamma, K_2(k(\mathbb{G}_m^2)))$. To split this map Γ-equivariantly, we would need to see that the map $\mathbb{Z}'[-4] \to Y_0 = H^{2,2}(\mathbb{G}_m^2)[-2]$ is trivial. Using the formula [Ram18, Proposition 4.9] for group cohomology of an amalgamation, that $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$, and that we have inverted 6, we find that $H^i(\Gamma, M) = H^i(\mathrm{SL}_2(\mathbb{Z}), M)^{C_2} = 0$ for $i > 1$. Thus,

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}[1/6])^{B\Gamma}}(\mathbb{Z}'[-4], H^{2,2}(\mathbb{G}_m^2)[-2]) = H^2(\Gamma, H^{2,2}(\mathbb{G}_m^2)) = 0.$$

Therefore, the map from Lemma 3.3 splits on the level of X_0, and we get a Γ-equivariant map $\rho : \mathbb{Z}'[-3] \to H^{2,2}(\mathbb{G}_m^2)[-2]$. We come to the main theorem of this section:

Theorem 3.4. *The class $\rho \in H^1(\Gamma, H^{2,2}(k(\mathbb{G}_m^2))[1/6])$ is a lift of the class representing Θ constructed earlier.*

Proof. We will construct the following commutative diagram with exact rows

[2]This is probably not quite the derived category of $\mathbb{Z}[1/6][\Gamma]$-modules, but is close to it, and embeds into it by a Schwede-Shipley argument.

$$
\begin{array}{ccccccc}
Y_0 = H^{2,2}(\mathbb{G}_m^2)[-2] & = & H^{2,2}(\mathbb{G}_m^2)[-2] & & & & \\
\downarrow & & \downarrow & & & & \\
X_0 = H^{*,2}(\mathbb{G}_m^2\backslash\{1\}) & \longrightarrow & H^{2,2}(k(\mathbb{G}_m^2))[-2] & \longrightarrow & X_1[1] & & \\
\downarrow & & \downarrow & & \| & & \\
\mathbb{Z}'[-3] & \longrightarrow & \overline{K_2}[-2] & \longrightarrow & X_1[1] & \longrightarrow & \mathbb{Z}'[-2] \\
 & & \| & & \downarrow & & \downarrow \\
 & & \overline{K_2}[-2] & \longrightarrow & \bigoplus_D k(D)^\times[-2] & \longrightarrow & \bigoplus_x \mathbb{Z}'[-2].
\end{array}
$$

Let's explain where this diagram comes from. The top left corner is the factorization $Y_0 \to X_0 \to H^{2,2}(k(\mathbb{G}_m^2))[-2]$. The group $\overline{K_2}$, defined simply as the quotient $H^{2,2}(k(\mathbb{G}_m^2))/H^{2,2}(\mathbb{G}_m^2)$, fits into a fiber sequence as pictured, and the map $\mathbb{Z}'[-3] \to \overline{K_2}[-2]$ is precisely the image of ρ under the quotient, which we wish to show is Θ. Since the top left rectangle has rows which are fiber sequences, it is a homotopy pushout (= homotopy pullback), and this gives us exactness of the middle two rows. There is an equivalence $\overline{K_2}[-2] \simeq Y_1[1]$, and the map to X_1 agrees with the one defined previously. The map $X_1[1] \to \bigoplus_D k(D)^\times[-2]$ is the canonical map arising as $\mathrm{cofib}(X_2 \to X_1)[1]$. The right-hand bottom rectangle is a map of short exact sequences of Γ-modules now, so the map $\mathbb{Z}'[-2] \to \bigoplus_x \mathbb{Z}'[-2]$ is uniquely determined. The cofiber of $X_1[1] \to \bigoplus_D k(D)^\times[-2]$ is given by $X_2[2] = \bigoplus_{x\neq 1} \mathbb{Z}'[-2]$, so the final map on the right-hand-side is nothing but inclusion onto the $\mathbb{Z}'[-2]\cdot 1$ summand.

The extension class in $H^1(\Gamma, \overline{K}_2)$ that ρ lifts is the short exact sequence given by the rightmost sequence (well strictly speaking shifted down to live in the heart) of the second to last row in the diagram above. This is the same as the extension on the very bottom pulled back along $\{1 \in \mathbb{G}_m^2\} : \mathbb{Z}' \to \bigoplus_x \mathbb{Z}'$, which is precisely how the class of Θ was defined to begin with, proving the claim. □

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