# D-MODULES AND CRYSTALLINE D-MODULES

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## 1. Standard Disclaimers

This is a brief note to accompany my talk for Math 216. We will discuss the geometric approach to de Rham cohomology expanding on the discussion from Bhatt chapter 2, the ideas originally due (in characteristic zero) to Gaitsgory-Rozenblyum.

## 2. Characteristic 0

To begin, let $k$ be a field of characteristic zero. Given a smooth $k$-scheme $X$, one would like to understand the de Rham cohomology of $X$, possibly with coefficients, whatever these coefficients should be. The point of this note is to explain a stacky approach to determining what "coefficients" for a cohomology theory means. Let's start with a definition:

**Definition 1.** Suppose that $X$ is a smooth $k$-scheme. The *de Rham stack* $X^{dR}$ is the stack with $R$-valued points $X^{dR}(R) = X(R_{red})$, where $R_{red}$ is the quotient of $R$ by its nilradical.

**Remark 1.** We mention the general underlying construction used above, the uninterested reader can safely skip this remark on a first reading. If we have some theory f, e.g., "f = de Rham," and we want to define "f-ification," there is a general procedure which Bhatt terms *transmutation*. The strategy proceeds by first defining what the f-ification of the affine line $\mathbb{G}_a$ is, specified by some ring stack $\mathbb{G}_a^f$ (over some fixed base ring/stack $R$). Transmutation is the process by which we assign to a scheme $X$ the stack with $S$-valued points $X^f(S) := \mathrm{Hom}_{/R}(\mathrm{Spec}(\mathbb{G}_a^f(S)), X)$, interpreted in the sense of derived algebraic geometry. For all the stacks appearing in this note, we will have some quasi-ideal $I \to \mathbb{G}_a$ on the ring scheme $\mathbb{G}_a$[1], which is enough to determine a ring structure on $\mathrm{cone}(I \to \mathbb{G}_a)$, making it into a ring stack. The above case takes $I = \widehat{\mathbb{G}_a} = \mathrm{Spec}(k[[t]])$.

**Proposition 1.** *For $X/k$ smooth, the cohomology of the structure sheaf on $X^{dR}$ computes the de Rham cohomology of $X$, that is:*

$$R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) \simeq R\Gamma(X, \Omega^\bullet_{X/k}) \in \mathcal{D}(k).$$

*Proof.* See Bhatt's lecture notes, or work this out explicitly as an exercise. □

In this way, we can think of vector bundles on $X^{dR}$ as "coefficients for de Rham cohomology," and if $E$ is a vector bundle on $X^{dR}$, we can define $H^i(R\Gamma(X^{dR}, E)) := H^i_{dR}(X, E)$. Let's be explicit about what this means by working with affine space (the general case is an exercise using the model set forth in what follows)

*Date*: mm/dd/yyyy.

[1] That is to say, $I$ is some a $\mathbb{G}_a$-module structure, such that the morphism $d : I \to \mathbb{G}_a$ is $\mathbb{G}_a$-linear and for $x, y \in I$, $(dx) \cdot y = (dy) \cdot x$.

**Lemma 1.** *Vector bundles on $(\mathbb{A}^n_k)^{dR}$ are equivalent to pairs $(E, \nabla)$, where $E$ is a vector bundle on $\mathbb{A}^n_k$ and $\nabla : E \to E \otimes_{\mathcal{O}_{\mathbb{A}^n}} \Omega^1_{\mathbb{A}^n/k}$ is a flat connection on $E$.*

*Proof.* Note that $\mathbb{A}^n$ is an fpqc cover of $(\mathbb{A}^n)^{dR}$, with $\mathbb{A}^n \times_{(\mathbb{A}^n)^{dR}} \mathbb{A}^n \simeq \mathbb{A}^n \times \widehat{\mathbb{A}^n}$. By fpqc descent for vector bundles, we can determine $\mathrm{Vect}((\mathbb{A}^n)^{dR})$ as the limit in categories of the first steps in the Cech nerve:

$$\mathrm{Vect}(\mathbb{A}^n) \rightrightarrows \mathrm{Vect}(\mathbb{A}^n \times \widehat{\mathbb{A}^n}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathrm{Vect}(\mathbb{A}^n \times \widehat{\mathbb{A}^n} \times \widehat{\mathbb{A}^n}).$$

The data of a vector bundle on the de Rham stack thus corresponds to a vector bundle $E$ on $\mathbb{A}_n$, an isomorphism

$$E \otimes_{k[x_1,...,x_n],a} k[x_1, ..., x_n][[t_1, ..., t_n]] \xrightarrow{\sim} E \otimes_{k[x_1,...,x_n]} k[x_1, ..., x_n][[t_1, ..., t_n]],$$

where $a$ denotes the fact that the map $k[x_1, \dots, x_n] \to k[x_1, \dots, x_n][[t_1, \dots t_n]]$ takes $x_i$ to $x_i + t_i$, satisfying some compatibility.

Such an isomorphism can equivalently be viewed as the data of a map

$$\psi : E \to E \otimes_{k[x_1,...,x_n]} k[x_1, ..., x_n][[t_1, ..., t_n]]$$

such that $\psi(x_i e) = \psi(e) \cdot (x_i + t_i)$, and $\psi(e) = e \otimes 1$ modulo $(t_1, ..., t_n)$. We may write $\psi(e) = \sum_I \psi_I(e) \otimes t^I$, using multi-index notation. Suggestively, for the multi-index which is 1 in degree $i$, we will call this index $\partial_i$, and multiplication of these partial derivatives will denote sums of these multi-indices. The compatibility here says that $\psi_0(e) = e$, and

$$\sum_I \sum_J \psi_I(\psi_J(e)) \otimes t^I y^J = \sum_I \psi_I(e) \otimes (t + y)^I. \tag{2.1}$$

This shows in particular that (since we are in characteristic zero), $\psi_I$ is completely determined by $\psi_{\partial_i}$ for $i = 1, ..., n$.

Define a connection $\nabla : E \to E \otimes_{k[x_1,...,x_n]} \Omega^1_{k[x_1,...,x_n]/k}$ by the formula $e \mapsto \psi_{\partial_i}(e) \otimes dx_i$.[2] Note that $\psi_{\partial_i} = \nabla(\partial_i)$, which is why we have chosen this notation. This is indeed a connection, as $\psi(x_i e) = \sum_I \psi_I(e) \otimes t^I (x_i + t_i) = e \otimes t_i + \sum x_i \psi_I(e) \otimes t^I$, so $\nabla(x_i e) = e \otimes dx_i + x_i \nabla(e)$, as claimed. Equation 2.1 says that $\psi_{\partial_i}(\psi_{\partial_j}(e)) = \psi_{\partial_i \partial_j}(e) = \psi_{\partial_j}(\psi_{\partial_i}(e))$, which is precisely the condition that the connection is flat. One can trace back through this proof to recover $\psi$ from a given flat connection. □

So, in characteristic zero, we get what we expect for coefficients for de Rham cohomology in analogy with differential geometry. It should be stressed that the assumption $k$ has characteristic zero is absolutely crucial above: for concreteness, if we take $n = 1$, then $\psi$ is the data of

$$\psi(e) = \sum_{i \geq 0} \frac{1}{i!} \nabla(\partial_x)^i(e) \otimes t^i.$$

[2] We are sending $t_i$ to $dx_i$ and ignoring $(t_1, ..., t_n)^2$.

## 3. The Characteristic p Situation

In this section we will restrict to working over a perfect field of characteristic $p > 0$. What goes wrong when we try to run the same machinery from above?

**Definition 2.** For $X/k$ a finite type qcqs scheme, define the *naive de Rham stack* as before: $X^{ndR}(R) = X(R_{red})$.

**Exercise.** Using fpqc descent for cohomology, show that $H^0(R\Gamma(\mathbb{G}_a^{dR}, \mathcal{O}_{\mathbb{G}_a^{dR}})) = k$. (Hint: setup the Cech nerve similar to the start of Lemma 1, though with cohomology groups in place of the category of vector bundles.)

If we were in characteristic zero, this would be all well in good, since there we live in the fairy tale land where $H^0_{dR}(\mathbb{A}^1_k) \simeq k$. This is not true in positive characteristic, as $dx^p = px^{p-1}dx = 0$, for instance.[3] Working on the $\mathbb{G}_a$-case, vector bundles on $\mathbb{G}_a^{ndR}$ are identified with vector bundles $E$ on $\mathbb{G}_a$ equipped with $\psi : E \to E \otimes_{k[x]} k[x][[t]]$ satisfying $\psi(xe) = \psi(e)(x + t)$, $\psi(e) = e \bmod t$, and 2.1. Rewriting 2.1 with $\psi(e) = \sum_{i \geq 0} \psi_i(e) \otimes t^i$, we get that

$$\sum_{i \geq 0} \sum_{j \geq 0} \psi_i(\psi_j(e)) \otimes t^i y^j = \sum_{i,j} \psi_{i+j}(e) \otimes \binom{i+j}{i} t^i y^j.$$

This says that not only do we have the data of a flat connection $\nabla$ with $\nabla(\partial_x)^p = 0$, but we also have a system of "formal divided powers" for $\nabla$, so vector bundles on $\mathbb{G}_a^{ndR}$ are already fairly complicated.

So, what can we do to improve the situation? If we naively use power series in our definitions above, we end up needing the data of divided powers of $\nabla$. One idea is to move these divided powers from $\nabla$ to the powers of $t$. Explicitly, let $\mathbb{G}_a^\sharp = \mathrm{Spec}(\mathbb{Z}[t, t^2/2!, \ldots, t^n/n!, \ldots] \otimes_{\mathbb{Z}} k)$ be the divided power envelope of $\mathbb{G}_a$. This is no longer a finite type $k$-scheme, but gives us a well-defined quasi-ideal of $\mathbb{G}_a$. We make the following definition:

**Definition 3.** The stack $\mathbb{G}_a^{dR}$ is the ring stack $\mathrm{Cone}(\mathbb{G}_a^\sharp \to \mathbb{G}_a)$. The de Rhamification of a general $k$-scheme $X$ is defined via transmutation.

One can show that whenever $X$ is smooth, $R\Gamma(X^{dR}, \mathcal{O}_{X^{dR}}) \simeq R\Gamma(X, \Omega^\bullet_{X/k}) \in \mathcal{D}(k)$, as we would hope. Before discussing coefficients, we need to introduce a new kind of curvature: $p$-curvature.

If $D$ is a derivation on a scheme $X/k$, then since $D^p(xy) = \sum_i \binom{p}{i} D^i(x) D^{p-i}(y) = D^p(x)y + xD^p(y)$, $D^p$ is also a derivation. The $p$-curvature of a connection $\nabla$ is the measure of the failure of $\nabla(D)^p = \nabla(D^p)$. Classical $D$-modules correspond to vector bundles with flat connection and zero $p$-curvature, but it turns out these will not quite be sufficient for our purposes. To read more about $p$-curvature, see §5 of Katz.

Vector bundles on this stack no longer identify with $D$-modules as in Lemma 1, but rather with so-called crystalline $D$-modules, to be explained in the talk:

[3]The Cartier isomorphism provides a very clean connection between the Frobenius and de Rham cohomology in characteristic $p$, generalizing what one sees on $\mathbb{A}^1$.

**Lemma 2.** *When $X/k$ is smooth, vector bundles on $X^{dR}$ identify with pairs $(E, \nabla)$ with $E$ a vector bundle on $X$, $\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$ is a flat connection with nilpotent $p$-curvature.*

*Proof.* We handle only the case of $\mathbb{A}^n$, leaving the general case to the reader (for general $X$, use the *pd*-envelope of the small diagonals $X \to X \times X$, $X \to X \times X \times X$ to get a similar story). Denoting by $\mathbb{A}^{n,\sharp} := (\mathbb{G}_a^\sharp)^{\times n}$, $\mathrm{Vect}((\mathbb{A}^n)^{dR})$ is the limit of

$$\mathrm{Vect}(\mathbb{A}^n) \rightrightarrows \mathrm{Vect}(\mathbb{A}^n \times \mathbb{A}^{n,\sharp}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathrm{Vect}(\mathbb{A}^n \times \mathbb{A}^{n,\sharp} \times \mathbb{A}^{n,\sharp}).$$

For a multi-index $I = (i_1, ..., i_n)$, let's write $t^{(I)}$ for the divided power element $\frac{t_1^{i_1} \dots t_n^{i_n}}{i_1! \dots i_n!}$[4]. As in Lemma 1, a vector bundle on the de Rham stack corresponds to a vector bundle $E$ on $\mathbb{A}^n$ equipped with a map

$$\psi : E \to E \otimes_{k[x_1,...,x_n]} k[x_1, ..., x_n][t^{(I)}]$$

$\psi(e) = \sum_I \psi_I(e) \otimes t^{(I)}$, $\psi_0(e) = e$. The compatibility criterion (2.1) now takes on the form

$$\sum_I \sum_J \psi_I(\psi_J(e)) \otimes t^{(I)} y^{(J)} = \sum_I \psi_I(e) \otimes (t + y)^{(I)} = \sum_{I,J} \psi_{I+J}(e) \otimes t^{(I)} t^{(J)}. \tag{3.2}$$

Again, everything is determined by $\psi_{\partial_j}$ for $j = 1, ..., n$. The formula $\nabla(e) = \sum_{j=1}^n \psi_{\partial_j}(e) dx_j$ defines a flat connection by essentially the same proof as lemma 1. The condition that $\nabla$ have nilpotent p-curvature translates to the fact that for some $m$, and all $I$ with $|I| > m$, we have $\psi_I(e) = 0$, which follows immediately from the tensor product defined above. $\square$

The actual talk will review a bit of the above, but focus more on the $D$-module perspective on these de Rham coefficients.

[4]The denominators of course aren't invertible in $k$ for $I >> 0$, this statement makes sense however since these generators are adjoined formally with these relations imposed