A BRIEF TOUR THROUGH FUKAYA CATEGORIES AND HOCHSCHILD HOMOLOGY

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Abstract. In this final project, we discuss some applications of Hochschild homology to the study of Fukaya categories, with a particular focus on the case of cotangent bundles. Many proofs will be omitted, opting to cite the source material (in part because reading all of the proofs for the information presented would take far too much time to get this in by the deadline), and when possible, we will aim to use the language of $\mathbb{E}_1$-algebras instead of $\mathbb{A}_\infty$-algebras, which serve as point-set models for the former. In §1, we will introduce some of the main players in this paper, and some of the main theorems we will discuss later. §2 will briefly describe the wrapped Fukaya category associated to a Liouville manifold (M, λ), after which we introduce the String topology category of Blumberg-Cohen-Teleman, and prove that it is equivalent to a module category over $\Sigma^\infty_+ \Omega M$ when M is a suitably nice manifold. Finally, §4 will give an overview of results on the Hochschild homology of these categories, and in §5, we will compute some examples.

§1. Introduction

Hochschild homology (over a base ring R) is an invariant associated to a $\mathbb{E}_1$-ring spectrum (or more generally a compactly generated[1] stable ∞-category) S that has recently found itself center stage in several areas of mathematics. From it one can obtain cyclic invariants, which have recently been used in p-adic geometry to define a new cohomology theory leading to the development of prismatic cohomology, and was also utilized in homotopy theory in the disproof of the only remaining conjecture of Ravenel- the telescope conjecture. Here we turn our focus to applications of Hochschild homology to the study of symplectic geometry by way of the (wrapped) Fukaya category. Below, we will introduce the wrapped Fukaya category in the context of what has been discussed in class. Abouzaid proves the following theorem:

Theorem 1.1 (Abouzaid, [**abouzaid2010**], Theorem 1.1)**.** *Let M be a Liouville manifold, $\mathcal{W}$ a full subcategory of its wrapped Fukaya category with a finite set of split generators, and let $\mathcal{B}$ be a full subcategory of $\mathcal{W}$ with finitely many objects. There is a map $\mathcal{OC} : \mathrm{HH}_*(\mathcal{W}) \to \mathrm{SH}^*(M)$, where $\mathrm{SH}^*(M)$ is the symplectic cohomology of M, such that if the image of the identity class $1 \in H^*(M) \to \mathrm{SH}^*(M)$ is in the image of the composite $\mathrm{HH}_*(\mathcal{B}) \to \mathrm{HH}_*(\mathcal{W}) \to \mathrm{SH}^*(M)$, then $\mathcal{B}$ split-generates $\mathcal{W}$.*

We will give a brief overview of this theorem in §4, in particular stating it a bit more properly, and then use it to compute some examples in §5.

This map $\mathcal{OC}$ can be realized on the categorical level in the special case that $M = T^*N$ is the cotangent bundle associated to a suitably nice manifold N. In this case, one can prove that there is an equivalence between $W(M)$ and the so-called string topology category $\mathcal{S}_N$ of N, which, when N is sufficiently nice, one can prove to be equivalent to the category of (left) modules over $\Sigma^\infty_+ \Omega N \otimes R$[2].

[1] Or using the machinery of Efimov, dualizable.

[2] In particular, this is one case where one would expect the Fukaya category to refine to a category equivalent to $\Sigma^\infty_+ \Omega N$-modules, that can be tensored with a commutative discrete ring to get back the ordinary Fukaya category over R.

There were many other papers that I wanted to include in this project, but due to time constraints, many had to be cut so that this would actually get done. As one example, Sheel Ganatra [Gan22] has studied how the open-closed map (on the level of chains) interacts with the S^1-equivariant structure on the source in target, allowing him to prove facts about the Hodge-de Rham degeneration in a purely symplectic fashion.

Conventions

- We fix a base ring R some field, say $\mathbb{Q}$ or $\mathbb{C}$, implicit in our notation. Hochschild homology will always be assumed to be taken over R.
- We freely use the language of Lurie [Lur17], and identify (triangulated) dg-categories with their stable enhancements.
- When discussing spectra, Σ^∞_+ is the (monoidal) functor taking a space to its suspension spectrum, and $\mathbb{S} = \Sigma^\infty_+ *$ is the unit for the smash product in the category of spectra, the sphere spectrum.
- Keeping with the above theme, we often identify a discrete commutative ring R with the associated Eilenberg-MacLane spectrum. Since modules over the $\mathbb{E}_\infty$−ring R are equivalent to the derived category $\mathcal{D}(R)$, we will often adopt the more homotopical notations, e.g., choosing to write $\Sigma^\infty_+ M \otimes_{\mathbb{S}} R \in \mathcal{D}(R)$ for the R-module $C^*(M, R)$ of singular chains on M.
- We will sometimes say category to mean ∞-category when the context is clear, and all tensor products are assumed to be derived unless otherwise specified.
- I generally like to have some claims I prove myself throughout, though these will certainly be able to be found in the literature. I will include a 🍃before any proof that I did as an exercise.

Fix (M, ω) a symplectic manifold with exact symplectic form ω.

Definition 2.1. We say that a vector field X on a symplectic manifold (M, ω) is **Liouville** if $\iota_X \omega = \lambda$ for some 1-form λ with $d\lambda = \omega$. We further say that M is a **Liouville manifold** if there is a compact submanifold $N \subseteq M$ with boundary ∂N such that $\lambda|_{\partial N}$ is a contact form, and $M \backslash N$ can be identified with $(1, \infty) \times \partial N$ with $\omega|_{M\backslash N}$ given by $d(r\lambda|_{\partial N})$, where r is the coordinate on $(1, \infty)$.

Example 2.2. Let N be a smooth closed oriented n-dimensional manifold, and consider the symplectic manifold $M = (T^*N, dp \wedge dq)$, where q is the coordinate on N and p the coordinate on the cotangent space at a point. Then, using the Liouville vector field $X = p\partial_p$, this manifold is Liouville with the form $\iota_X\omega = \iota_X(dp \wedge dq) = pdq$[3]. To see that this is Liouville, we can choose a Riemannian metric g on N (denote g^* the induced bilinear form on the cotangent space), take the closed unit disk bundle in $D \subseteq T^*N$ as our compact submanifold. The boundary ∂D is the unit sphere bundle, and taking $r = |p|$, we have that $M\backslash D \simeq \partial D \times (1, \infty)$ by scaling. (✐) Since this is an important point, let's check in some detail that $\lambda|_{\partial D}$ is a contact form: when we restrict to ∂D, our form, still takes the form pdq, and we have to check that $(pdq) \wedge (d(pdq)^{\wedge n-1})$ is nonzero. This is pulled back from pdq under $i : \partial D \to T^*N$. Pulling it back further under $\pi : T^*N\backslash D \simeq \partial D \times (1, \infty) \to \partial D$ gives the form $\frac{p}{|p|}dq$ (this form is also mentioned in [Coh15]). We can now note that, in local coordinates, we have that $\partial_{p_i}(g^*(p, p)) = \partial_{p_i} \sum_{k,j} g^{kj} p_k p_j = \sum_k 2g^{ki}p_k = \frac{2g^*(p,p_i)}{p_i}$ (and is zero at points where $p_i = 0$). So, $d\frac{1}{|p|}$ (which is differentiable on $T^*N\backslash D$) is equal to $-\frac{\sum_{i=1}^n g^*(p,p_i)}{p_i \cdot |p|^3} dp_i$ plus some terms involving Christoffel symbols of the first time which we ignore since they have dq_js in them which will vanish in the final tensor expression we are after (which if it has one dq_j from one of these has at least $n + 1$ terms of the form dq_k, so two must coincide and that term vanishes). We then have that $d(\frac{p}{|p|}dq) = \frac{1}{|p|}\omega - \sum_{j=1}^n \frac{\sum_{i=1}^n p_j g(p,p_i)}{p_i |p|^3} dp_i \wedge dq_j$, plus some terms attached to $dq_j \wedge dq_k$ (again ignored), and we note that the exterior product of this second expression with pdq is zero, since it is $-\sum_{j,k} \frac{\sum_{i=1}^n p_j p_k g(p,p_i)}{p_i |p|^3} dp_i \wedge dq_j \wedge dq_k$ and we can match up the terms for $j < k$ which cancel out those for $j > k$. Thus, we in fact have that $\pi^*(i^*(pdq \wedge (dp \wedge dq)^{\wedge n-1})) = \frac{1}{|p|^n}(pdq) \wedge \omega^{\wedge n-1}$, which is necessarily nonzero since we can multiply by $|p|^n$ to get $(pdq) \wedge \omega^{\wedge n-1}$, which has derivative $\omega^{\wedge n}$, a volume form. The last condition is clear.

Now, we recall from [Aur] how to define the wrapped Fukaya category $\mathcal{W}(M)$ associated to a Liouville manifold M. We proceed similarly to in class, defining $\mathcal{W}(M)$ to have objects exact Lagrangian submanifolds $L \subseteq M$ subject to the condition that they are invariant under the λ flow outside a compact set, with $\lambda|_L$ compactly supported. In the cotangent space example, the fiber over any point T_x^*N gives a Lagrangian submanifold, the flow associated to $p\partial_p$ is $\phi^t(p, q) = (e^t p, q)$, which in fact fixes T_x^*N entirely, and $pdq|_{T_x^*N} = 0$ since q is constant. Hamiltonian isotopy is now refined to enforce that our Hamiltonians must, outside a compact subset, take on the form $H : (1, \infty) \times \partial N \to \mathbb{R}$, $H(r, x) = r^2$. We then have that, as with the ordinary Fukaya category, maps between Lagrangians L and L' are given by intersection points of L and $\phi_H^1(L')$ (at least when the intersection is transverse).

When the intersection is not transverse, we may need to modify L by a Hamiltonian isotopy

[3]The original reference [Coh15] says that $-qdp$ is the Liouville form which I believe to be a typo.

such that the same conditions as before hold, and then work with that. For the modifications to the construction of the higher A_∞-operations, see [Aur].

We will take a particular interest at first in the conormal wrapped Fukaya category $\mathcal{W}^{con}(T^*N)$. This is the wrapped Fukaya category with objects $\mathcal{N}^*D$ for $D \subseteq N$ a closed oriented connected smooth submanifold, and $\mathcal{N}^*D$ the conormal bundle to D, i.e., $\mathcal{N}^*D = \{f \in T^*N : f|_D = 0\}$. We have:

Lemma 2.3. *The conormal bundle $\mathcal{N}^*D$ is a Lagrangian submanifold, fixed by the λ-flow, with $\lambda|_{\mathcal{N}^*D} = 0$.*

Proof. Let $D \subseteq N$ be as above. Then at any point, choose suitable local coordinates extending local coordinates on D so that D has coordinates $(q_1, \dots, q_r)$ and N local coordinates $q_1, \dots, q_r, q_{r+1}, \dots, q_n$. If the dual coordinates to these are given by $p_1, \dots, p_n$, then we have that $\mathcal{N}^*D$ has local coordinates $p_{r+1}, \dots, p_n$ at this point, and in particular, $\lambda|_{\mathcal{N}^*D} = \sum_i p_i \wedge dq_i|_{\mathcal{N}^*D} = 0$, which also implies that $\mathcal{N}^*D$ is a Lagrangian submanifold, as it has the correct dimension and $\omega = d\lambda$. Since at any point $q \in D$, $\mathcal{N}_q^*D \subseteq T_q^*N$ is a subspace, it is invariant under scaling, and in particular, $\mathcal{N}^*D$ is invariant under the λ flow, as described above, giving the claim. □

§3. The String Topology Category

For this section, we fix a connected, closed, oriented, n-dimensional smooth manifold N. Further assume that N is equipped with a collection $\mathcal{D} = \{D_i \subseteq N\}$ of smooth submanifolds called "D-branes", which will later be taken to be the collection of smooth connected oriented closed submanifolds of N. To any pair D_1, D_2 of D-branes, we associate the path space from D_1 to D_2:

Definition 3.1. Let D_1 and D_2 be two D-branes. The path space from D_1 to D_2 is $\mathcal{P}_{D_1,D_2} = \{\gamma : [0,1] \to N : \gamma(0) \in D_1, \gamma(1) \in D_2\}$.

For example, if D_1 and D_2 are both points, then $\mathcal{P}_{D_1,D_2}$ is (non-canonically) equivalent to ΩN, and if $D_1 = D_2 = N$, then $\mathcal{P}_{D_1,D_2}$ is the free path space of N, which is homotopic to N. To these data, Blumberg-Cohen-Telemen associate:

Definition 3.2. Let N, $\mathcal{D}$ be as above. The **string topology category** $\mathcal{S}_N^{\mathcal{D}}$ over a field k is a DG category which has objects the elements of $\mathcal{D}$, and the mapping spaces are given by $\mathrm{Hom}_{\mathcal{S}_N^{\mathcal{D}}}(D_1, D_2) \simeq H_{*-\dim D_1}(\mathcal{P}_{D_1,D_2}, k)$. The composition is defined using open-closed cobordisms and umkehr maps as in [CHV06] section 3.3.

Remark 3.3. One can also define the above category over spectra replacing $H_*(\mathcal{P}_{D_1,D_2}, k)$ with $\Sigma_+^\infty \mathcal{P}_{D_1,D_2}$ and defining the operations on the level of spectra as in [BCT09] 2.12.

The compositions arise from considering so-called open-closed cobordisms between D-branes, and using umkehr maps defined similarly to the Chas-Sullivan product on the homology of the free loop space LM of a manifold M (See [CHV06]).

Now, fix a basepoint $x_0 \in N$. We have that, for a D-brane D, the homotopy fiber of $D \to N$ is modeled by $\mathcal{P}_{x_0,D}$. We can use this to get the following commutative diagram, for any pair of D-branes D_1, D_2, where the bottom rectangle, bottom right square, right rectangle, and total square are homotopy cartesian, which implies every square in the diagram is as well:

$$\begin{array}{ccccc}
\Omega N & \longrightarrow & \mathcal{P}_{x_0,D_1} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{x_0,D_2} & \longrightarrow & \mathcal{P}_{D_1,D_2} & \longrightarrow & D_2 \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & D_1 & \longrightarrow & N.
\end{array}$$

In the case $D_1 = x_0$, we adopt the notation from [BCT09] and write $\mathcal{P}_{x_0,D} = F_D$, the homotopy fiber of $D \to N$. Blumberg-Cohen-Teleman implicitly use the following fact from homotopy theory (see [(ht])

Proposition 3.4. *Let $E \to F \to M$ be a homotopy fiber sequence. Then E has a ΩM-module structure, and $* \times_{\Omega M} E \simeq F$. In particular, this holds on the stable level and after passing to chains.*

□

From this, we conclude:

Lemma 3.5. *Assume that D has the homotopy type of a finite CW complex. Then $C_*(F_D)$ is a compact object in the category of $C_*(\Omega N)$-modules.*

Proof. (✐) From the equivalence in Proposition 3.4 and the fiber sequence $\Omega N \to F_D \to D$, we have that $C_*(\Omega N) \otimes_{C_*(\Omega D)} k \simeq C_*(F_D)$. Since tensor product sends compact objects to compact objects, it suffices to show that k is a compact $C_*(\Omega D)$-module. There is a canonical functor (abusively considering D as an ∞-groupoid), $F : D \to C_*(\Omega D) - \mathrm{Mod}$ taking the basepoint in D to $C_*(\Omega D)$. We have $\mathrm{colim}\, F \simeq k \otimes_{C_*(\Omega D)} C_*(\Omega) D \simeq k$, and since D has the homotopy type of a finite CW complex, this colimit is finite, so by Lurie, its colimit, k, is a compact $C * (\Omega D)$-module. □

Lemma 3.6. *There is an equivalence of k-modules $C_*(\mathcal{P}_{D_1,D_2}) \simeq C_*(F_{D_1}) \otimes_{C_*(\Omega N)} C_*(F_{D_2})$.*

Proof. We note as in [BCT09] that

$$C_*(F_{D_1}) \otimes_{C_*(\Omega N)} C_*(F_{D_2}) \simeq k \otimes_{C_*(\Omega D_1)} C_*(\Omega N) \otimes_{C_*(\Omega N)} C_*(F_{D_2})$$

$$\simeq k \otimes_{C_*(\Omega D_1)} C(F_{D_2}) \simeq C_*(\mathcal{P}_{D_1,D_2})$$

where we have used Proposition 3.4 applied to the fiber sequence $F_{D_2} \to \mathcal{P}_{D_1,D_2} \to D_1$ in the second step. □

With all this background in place, we recover:

Theorem 3.7. *The category $\mathcal{S}_N^{\mathcal{D}}$ is equivalent to the full subcategory of* $\mathrm{mod}(C_*(\Omega N))$ *consisting of the objects $C_*(F_D)$.*

Sketch. This uses the above computations to see that the functor $\mathrm{Hom}_{\mathcal{S}_N^*}(x_0, -)$ has the appropriate image, and that $\mathrm{Hom}_{C_*(\Omega N)}(C_*(F_D), C_*(F_{D'})) \simeq C_*(\mathcal{P}_{D,D'})$ by a dualizable object argument and Lemma 3.5. □

We care about the String topology category described above, since it connects back to the conormal Wrapped Fukaya category $\mathcal{W}^{con}(T^*N)$. In particular, we will investigate the following theorem:

Theorem 3.8. *Suppose that N is a connected, oriented, closed manifold. Then there is an A_∞-quasi-equivalence $\mathcal{W}^{con}(T^*N) \to \mathcal{S}_N^{\mathcal{D}}$ taking the conormal bundle of a submanifold D of N in $\mathcal{W}^{con}(T^*N)$ to the object represented by D in $\mathcal{S}_N^{\mathcal{D}}$.*

This was conjectured in Blumberg-Cohen-Teleman [BCT09] and later proven by Cohen-Ganatra in [Coh15]. The proof goes through constructing a Morse-theoretic model for the string topology category, and then using this to construct the desired quasi-equivalence.

§4. Hochschild Homology

Hochschild homology (in one formulation) can be thought of as the "universal target of the trace map." For us, it will be useful due to the following theorem proved by Abouzaid:

Theorem 4.1. *Let M be a Liouville manifold, $\mathcal{W}$ a full triangulated subcategory of its wrapped Fukaya category with a finite set of split generators, and let $\mathcal{B}$ be a full subcategory of $\mathcal{W}$ with finitely many objects. There is a map $\mathcal{OC} : \mathrm{HH}_*(\mathcal{W}) \to \mathrm{SH}^*(M)$, where $\mathrm{SH}^*(M)$ is the symplectic cohomology of M, such that if the image of the identity class $1 \in H^*(M) \to \mathrm{SH}^*(M)$ is in the image of the composite $\mathrm{HH}_*(\mathcal{B}) \to \mathrm{HH}_*(\mathcal{W}) \to \mathrm{SH}^*(M)$, then $\mathcal{B}$ split-generates $\mathcal{W}$.*

Here, by $\mathrm{HH}_*(\mathcal{B})$, we may take this to mean either an explicit chain complex as defined in [Gan22], or we may consider it abstractly as the Hochschild homology of $\mathrm{End}(\oplus_{L\in \mathrm{Ob}(\mathcal{B})} L)$ considered as an $\mathbb{E}_1$-R-algebra, using Morita invariance. We are abusing terminology in the statement and assuming that $\mathcal{W}$ is a triangulated subcategory of (twisted complexes) for the wrapped Fukaya category of M.

The strategy of proof proceeds by, for any $K \in \mathcal{W}$, constructing a bimodule A_K associated to K, from the Yoneda modules associated to K itself. Abouzaid constructs a diagonal map $\mathcal{B} \to A_K$, and then constructs the following commutative diagram, commuting up to signs:

$$\begin{array}{ccc} \mathrm{HH}_*(\mathcal{B}) & \longrightarrow & \mathrm{HH}_*(\mathcal{B}, A_K) \\ \downarrow & & \downarrow \\ \mathrm{SH}^*(M) & \longrightarrow & \mathrm{HW}^*(K, K) \end{array}$$

where $\mathrm{HW}^*(K,K) = \mathrm{End}_{\mathcal{W}}(K)$ is the wrapped Floer homology of K. The assumptions of the theorem imply that the identity class 1_K is in the image of the map from Hochschild homology of $\mathcal{B}$, and Abouzaid proves that, if this is the case, then $\mathcal{K}$ is in the full subcategory of objects split generated by $\mathcal{B}$. We can think of this statement through the heuristic "if the identity of an object K arises as a linear combination of traces of endomorphisms of objects in $\mathcal{B}$, then that object can be built out of objects of $\mathcal{B}$."

This theorem is useful to us, since it allows us to give moral justification for the following:

Theorem 4.2. *(Twisted modules over) The wrapped Fukaya category T^*M of a closed oriented connected smooth manifold M is equivalent to the category of $C_*(\Omega M)$-modules.*

"proof". From the results of §3, we know that the conormal wrapped Fukaya category $\mathcal{W}^{con}(T^*M)$ is a full subcategory of compact $C_*(\Omega M)$-modules, and in particular, is generated by the cotangent fiber $T^*_q M$ of any point $q \in M$. Nikolaus-Scholze [NS18] IV.3.3 prove that $\mathrm{THH}(\Sigma^\infty_+ \Omega M) \simeq \Sigma^\infty_+ LM$, and tensoring with R gives $\mathrm{HH}_*(C_*(\Omega M)) \simeq C_*(LM)$, where LM denotes the free loop space of M. By Morita theory, $\mathrm{HH}_*(\mathcal{W}^{con}(T^*M)) \simeq C_*(LM)$, which was shown in [Vit18] (see also [AS09]) to be equivalent to $\mathrm{SH}^*(M)$.[4] Due to this equivalence, the conditions of the theorem are satisfied, so taking any object $L \in \mathcal{W}(T^*M)$, we can take $\mathcal{W}$ in the statement to be the full subcategory split generated by L and $T_q M$, $\mathcal{B} = \{T_q M\}$, and then $\mathcal{OC}$ is an equivalence, such that L is in the full subcategory split generated by $T_q M$. Therefore $T_q M$ split generates $\mathcal{W}(T^*M)$, from which the claim follows. □

[4]"proof" since it is not necessarily obvious that $\mathcal{OC}$ witnesses this equivalence.

§5. ✐ Examples and Applications

In this section, we discuss examples related to the above constructions, and perform some computations. First, from the homotopical side, Theorem 3.8, allows us to construct a Liouville manifold M such that the wrapped Fukaya category of M, $\mathcal{W}(M)$, is equivalent to modules over A for various $\mathbb{E}_1$-R-algebras A. As a first example:

Example 5.1. Let A be the free $\mathbb{E}_1$-algebra over R on a class in degree $k > 0$. That is, $A = C_*(\Omega\Sigma S^k) = C_*(\Omega S^{k+1})$. Since the sphere S^{k+1} is connected, closed, and oriented, we have that $\mathrm{Mod}(A) \simeq \pi(\mathrm{Tw}(\mathcal{W}(T^*S^{k+1})))$. A standard Serre spectral sequence shows that $C_*(\Omega S^{k+1}) \simeq R[x]$ is a polyomial algebra on a class x with $|x| = k$.

Let's try to see that the cotangent fiber over a point has the expected endomorphism algebra in this wrapped Fukaya category:
This would be very hard to attempt directly with coordinates, even for T^*S^2. Using the standard Riemannian metric, the Hamiltonian $H(p,q) = g^*(p,p)^2$ gives rise to the flow ϕ^t_H. In local coordinates $(\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta))$, the Riemannian metric is given by $\sin^2(\theta)d\varphi^2 + d\theta^2$. The differential equations $\phi^t_H = (d\varphi(t), d\theta(t), \varphi(t), \theta(t))$ must satisfy are therefore ($d\varphi$ and $d\theta$ the cotangent coordinates):

$$\partial_t(d\varphi(t)) = 0; \partial_t(d\theta(t)) = \frac{\cos(\theta(t))}{\sin^3(\theta(t))}\varphi(t)^2; \partial_t(\varphi(t)) = \frac{d\varphi(t)}{\sin^2(\theta(t))}; \partial_t(\theta(t)) = d\theta(t).$$

We need a better way to figure out what this flow should be, and coming to our rescue is the notion of a cogeodesic flow [Hil21][5]. For our Riemannian manifold M, the time $t = 1$ flow can be described by the exponential map together with the isomorphism determined by the metric: $T^*M \xrightarrow{\sim} TM \xrightarrow{exp} TM \xrightarrow{\sim} T^*M$, (at least assuming that the exponential exists for enough time). However, there is a problem. If we are working on the spheres S^n, with both Lagrangians the cotangent fiber at the same point q, $T^*_qS^n \cap \phi^1_H(T^*_qS^n)$ is not a transverse intersection. Indeed, any intersection which is not at the zero section will have connected component a whole S^{n-1}. Now, one may say that this is absolutely horrible, horrid, another example of horrible behavior of our spheres S^n for $n \geq 2$. Alas, the fix is rather simple, and this behavior is actually our saving grace for determining the grading on the Hom complex. To correct the issue, simply take local coordinates $(p_1, ..., p_n, q_1, ..., q_n)$, and then let $H : T^*S^n \to \mathbb{R}$, $H(p,q) = p_i$. The associated Hamiltonian vector field to H is $X_H = \partial_{q_1}$, so under the flow of H, $T^*_qS^n$ maps to $T^*_sS^n$ for some close enough point s to q, which we can use as our perturbed version L' of $L = T^*_qS^n$ to construct the Floer complex. Now, the intersection $\phi^1_H(L) \cap L'$ is transverse, and is $\mathbb{Z}$ as a set.

This intersection corresponds to Hamiltonian trajectories for H from L to L', which further corresponds to geodesic flows from q to s. This can be explicitly seen from the above by noting the intersections are identified with the points r in the tangent space with $\exp_q(r) = s$, which are precisely the geodesic flows from q to s. It is a classical fact[6] that the index of the intersection point in the sense of Floer theory is negative the Morse index of the geodesic considered as a critical point of the energy funtional (see [Ye20] for the definitions of this part). Thus, in our S^n case, we know from our Riemannian geometry classes that these geodesics lie

[5]The S^1 example is where it is easiest to see the connection Hamiltonian flow as connected to the exponential map, and in general describing it in local coordinates using our Riemannian metric, it is not hard to show that the Hamiltonian flow we are after is the cogeodesic flow.

[6]c.f. I'm taking this on faith, but it sounds believable.

on great circles through q and s. Earlier, we said that the fact that the intersections of L with $\phi_H^1(L)$ being spheres of one dimension lower would be our saving grace, and with the help of the index theorem, Theorem 7.4 of [Ye20], we can now see why. The index theorem says that $\operatorname{ind}(g)$ = number of conjugate points with multiplicity of $g(t)$, for our given geodesic g. The conjugate points for our geodesic are precisely the points v where $D\exp_q(v)$ (writing exp instead of g here for clarity, opting to use Petersen's notation) is singular, which are precisely the points where $g(t) = q$ and $t > 0$, or $g(t) = -q$, the point antipodal to q. The multiplicity of these points is the dimension of the kernel of $D\exp_q$ at these points. As we essentially noted via the "bad behavior," at these conjugate points, there is a whole S^{n-1} worth of points mapping to it, such that the index is $\dim(S^{n-1}) = n - 1$.

With all this in hand, determining the grading is easy. We can simply count the number of conjugate points. The shortest path from q to s does not pass any, so it has (Morse, which we will say for all indices for now, so we are working with - the Floer index) index 0, and we denote it by 1. There is a geodesic which "flees away from" s, passing through $-q$ and hitting s, not covering the great circle through q and s. It's only critical point is $-q$ of index $n - 1$, so that this curve has index $n - 1$, we denote this generator by x. Concatonating m loops passing through $s, -q, -s, q$ at s to the first path, we get geodesics passing through q and $-q$ m times, so these have index $2m(n - 1)$, and concatonating m loops going the other direction to the second geodesic, we get geodesics with index $(2m + 1)(n - 1)$. For $n > 2$ there are no differentials for degree reasons, and for $n = 2$, one can check that μ^1 vanishes on x, so will vanish identically by what we are about to prove. To figure out the multiplication on x, we perform a trick. Replace the point s with a point close to $-q$ but not $-q$, and add in a third point r (strictly) between $-s$ and $-q$ on the same great circle as q and s, close to $-s$. The geodesic represented by x from q to s and the geodesic represented by x from s to r can be concatonated to give a geodesic from q to r, passing through q and $-q$, which is exactly our class x^2 in degree $2(n - 1)$. While one needs to check a few other triangles in the definition of μ^2 cannot happen to make it formal, this argument (applied repeatedly) at least makes it intuitively clear why $H_* \operatorname{Hom}(T_q^* S^n, T_q^* S^n) = \mathbb{Z}[x]$ with $|x| = n - 1$, from the point of view of Morse theory.

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