A PBW THEOREM IN THE VERLINDE CATEGORY

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Abstract. This is a copy where I make changes I wanted to after submitting.
This note discusses the universal Verlinde categories as introduced in [Ost15], and in particular the theory of Koszul complexes and Lie Algebras in the Verlinde category, following [Eti16]. The first section we will introduce some background and then define the Verlinde category. After that we will give some background about Koszul complexes and Koszul algebras. In third section, we introduce operadic Lie Algebras and the definitions of Lie algebras and the PBW condition in an arbitrary symmetric tensor category. Finally, in the fourth section, we will prove results about Lie algebras and Koszulity in the Verlinde category.

Many proofs will be omitted and their sources cited instead for brevity. A proof will be marked with 🍃 if it is my own proof, along with a reference linked next to it to where a proof was given by somebody else, if I can find one. Throughout for sums and operations in abelian categories we will use the notation as in [Kra21], and some sums may be implicitly assumed to take place in the category of ind-objects (primarily when we are working with those) in order to ensure that necessary coproducts exist.

1. The Verlinde Category

Let k be an algebraically closed field. Following ([EGNO16]), and the definition given in [Ost15], a k-linear pre-Tannakian category $\mathcal{C}$ is defined to be a k-linear symmetric tensor category (in the sense of [EGNO16]) which is essenitally small, locally finite dimensional[1], and all objects are of finite length. A pre-Tannakian category is called a symmetric fusion category when it is semisimple with a finite number of isomorphism classes of simple objects([Ost15], 1.3). A k-linear tensor category is said to have a spherical structure if the conditions of ([Ost15], 2.4) are satisfied, in particular, that left and right quantum trace of an endomorphism are defined and agree. In such a case, one defines the dimension of an object X as the trace of the identity morphism. Due to ([EGNO16], 4.7.3), dim determines a ring homomorphism from the Grothendieck ring $K(\mathcal{C})$ of a spherical k-linear fusion category $\mathcal{C}$ to k. Ostrik uses this to define a notion of global dimension for $\mathcal{C}$, calling $\mathcal{C}$ non-degenerate if it's dimension is nonzero, and proved the result

Proposition 1 ([Ost15], 2.9). *Given a spherical fusion category $\mathcal{C}$ with $K(\mathcal{C}) \otimes k$ semisimple, then $\mathcal{C}$ is non-degenerate.*

$\square$

Using a similar method, we can derive the following result, included as a brief tangent[2]

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[1]That is, $\mathrm{Hom}(a, b)$ is finite dimensional as a k-vector space for all $a, b \in \mathcal{C}$.

[2]I do not know a reference for what follows, surely it has been published somewhere but I do not think I know the right keywords for it, and just came up with this after looking at the proof of the above Proposition.

Proposition 2 (✐). *For any k-linear spherical fusion category $\mathcal{C}$, if $K(\mathcal{C})$ is the Grothendieck ring of $\mathcal{C}$, then $K(\mathcal{C}) \otimes_{\mathbb{Z}} k$ is a Frobenius algebra.*

Proof. We prove that $K(\mathcal{C}) \otimes k \simeq \mathrm{Hom}_k(K(\mathcal{C}) \otimes k, k)$ as left $K(\mathcal{C}) \otimes k$-modules, which is implied by $K(\mathcal{C}) \otimes k$ being finite dimensional and the existence of a linear functional $l : K(\mathcal{C}) \otimes k \to k$ whose kernel contains no proper left ideal ([Rot08] chp 4). Note that $K(\mathcal{C}) \otimes k$ is finite dimensional over k, and define l by $1 \mapsto 1$, and $[X] \mapsto 0$ for all $[X] \in \mathcal{O}(\mathcal{C})\backslash\{1\}$ (where $\mathcal{O}(\mathcal{C})$ is the set of isomorphism classes of simple objects of $\mathcal{C}$). We see that if $\sum_{[X]\in\mathcal{O}(\mathcal{C})} b_{[X]}[X]$ is in the kernel, and nonzero, then we can take $T := \oplus_{[X]\in\mathcal{O}(\mathcal{C})} X^{\oplus b_{[X]}} \in \mathcal{C}$. From here, we see that, for some $b_{[X]}$ nonzero, $\dim \mathrm{Hom}_{\mathcal{C}}(X^* \otimes T, 1) = \dim \mathrm{Hom}_{\mathcal{C}}(T, X) = b_{[X]}$, so that the image of $[X^*][T]$ under l is nonzero, and hence $\ker l$ contains no proper nonzero left ideal, and hence $K(\mathcal{C}) \otimes k$ is a Frobenius algebra. □

Note that in the proof, we have implicitly used the fact that k is algebraically closed so that there are no nontrivial finite dimensional division algebras over k, which ensures $\mathrm{Hom}_{\mathcal{C}}(X, X) \simeq k$ as k-algebras when X is a simple object in $\mathcal{C}$.

Now, for a symmetric (necessarily spherical) pre-Tannakian category $\mathcal{C}$, we define negligible morphisms as in ([Ost15], 2.5), namely, morphisms $f : X \to Y$ such that for all $g : Y \to X$, fg has trace zero. These morphisms form a tensor ideal ([Ost15], 2.5), so we can take the quotient $\bar{\mathcal{C}}$, and we prove

Proposition 3 (✐, [EGNO16] Exercise 8.18.9; [Ost15] Proposition 2.13). *If $\mathcal{C}$ is as above, then $\bar{\mathcal{C}}$ is semisimple, with simple object those isomorphic to $\bar{X}$ where X is an indecomposable object of $\mathcal{C}$ with* $\dim(X) \neq 0$.

Proof. Suppose $\mathcal{C}$ is as above. First, for any $Y \in \mathcal{C}$, we can write Y as a direct sum of indecomposable objects, and since we are quotienting by an ideal, this will still hold in $\bar{\mathcal{C}}$, so it suffices to show that if X is indecomposable, $\bar{X}$ is either simple or zero. For any X indecomposable, we note that $R_X := \mathrm{End}_{\mathcal{C}}(X)$ is a finite dimensional k-algebra, hence both Noetherian and Artinian, and it contains no nontrivial idempotents, or else X would not be indecomposable. For any element $f \in \mathrm{rad}(R_X)$, f is nilpotent, which can be seen since if $f : X \to X$ is not an isomorphism, by the finite length assumption, $\mathrm{im}\, f^n = \mathrm{im}\, f^{n+1}$ for some n, and then necessarily, f descends to an isomorphism g on $\mathrm{im}\, f^n$, from which we see that, if $i : \mathrm{im}\, f^n \to X$ is the canonical inclusion, and p the canonical projection such that $ip = f^n$, we can write $g = pfi$, and then $ig^{-1}pf : X \to X$ is such that $ig^{-1}pfig^{-1}pf - ig^{-1}gg^{-1}pf = ig^{-1}pf$ is an idempotent, so that $ig^{-1}pf = 0$ since f is not an isomorphism and X is indecomposable. From this, we find that, since i is monic, and g^{-1} an isomorphism, $pf = 0$, and since $0 = ipf = f^{n+1}$, f is nilpotent. Now, by ([EGNO16] 4.7.5), and induction, $0 \to \mathrm{im}\, f^{s+1} \to \mathrm{im}\, f^s \to \mathrm{coker} \to 0$, f has zero trace on the first factor and by construction on the third, so it has zero trace on $\mathrm{im}\, f^s$, repeating this downwards, we see f has zero trace on X. Hence, $\mathrm{End}_{\bar{\mathcal{C}}}(X) \simeq R_X/(\mathrm{rad}(R_X)) \simeq k$ when $\mathrm{Tr}(1_X) \neq 0$, since the trace then is zero only on the radical, a two sided ideal, and having endomorphism ring a division ring implies X is simple. On the other hand, if $\mathrm{Tr}(1_X) = 0$, then since every element of R_X can be written as $a1_X + t$ with $a \in k$ and $t \in \mathrm{rad}(R)$, along with the fact trace is k-linear, we see that $\bar{X} = 0$ in $\bar{\mathcal{C}}$. The claim is shown. □

Now we are ready to define the Verlinde category Ver_p. If we have the category $k[C_p] - mod$ of finitely generated left $k[C_p]$ modules where C_p is the cyclic group of order p, let the Verlinde category Ver_p be the category obtained by quotienting out $k[C_p] - mod$ by negligible morphisms as above ([Ost15], 3.2).

2. Koszul Complexes and Koszul Algebras

Let $\mathbb{K}$ be an algebraically closed field as before, and let $\mathcal{C}$ be a symmetric $\mathbb{K}$-linear tensor category. For an object V of $\mathcal{C}$, we define the algebras SV and ΛV as in [EHO15], namely, the graded algebra SV, the symmetric algebra of V, has n^{th} component $V^{\otimes n}/(\sum_{\sigma\in S_n}\mathrm{im}((1-\sigma) : V^{\otimes n}\to V^{\otimes n})$, where σ by abuse also stands for the image of the obvious map it induces on $V^{\otimes n}$; and the exterior power algebra ΛV is defined as the graded algebra obtained from dualizing comultiplication on the exterior algebra of V^*, the quotient of the tensor algebra of V^* having n^{th} graded component $(V^*)^{\otimes n}/(\sum_{\tau\in S_n \text{ a transposition}}\mathrm{im}((\tau+1) : (V^*)^{\otimes n}\to (V^*)^{\otimes n})$. With these in hand, we define the Koszul complex associated to V as the chain complex $K^\bullet(V) = SV\otimes\Lambda^\bullet V$[3] with differential defined as in ([Eti16], Definition 2.1). As in [LV12], we note that SV is an augmented algebra, so that we can write $SV\simeq 1\oplus \bar{SV}$, where $\bar{SV}$ is called the reduced symmetric algebra. This makes 1 into an SV-module in the obvious way, and in particular, we say the Koszul complex $K^\bullet(V)$ is exact ([Eti16], def 2.2) if it has homology isomorphic to 1 in degree 0, and 0 elsewhere. We then have

Definition 1 ([Eti16], Definition 2.3)**.** We say that $V\in\mathcal{C}$ is a Koszul object when $K^\bullet(V)$ is exact in the sense of the previous definition.

We will say a bit more about this category of modules for an ind-algebra A, since I have not been able to find any sources on this and it seems important, so we will elaborate on this, going off of remark 2.2 and Definition 2.3 in [Eti16]. First, let's define an ind-algebra A to mean an ind-object which is of the form $1\oplus\bar{A}$ where 1 lives in degree zero, and $\bar{A}$ is a positively graded non-unital algebra in the category of ind-objects of $\mathcal{C}$ with the induced tensor product such that all of its homogeneous components are objects of $\mathcal{C}$. With this in hand, we will talk about the category of A-modules defined with objects those ind-objects X together with a map $\bar{m_X} : \bar{A}\otimes X\to X$ (or equivalently $m_X : A\otimes X\to X$ such that the induced $1\otimes X\to X$ is the canonical isomorphism), which renders the diagram

$$\begin{array}{ccc} A\otimes A\otimes X & \xrightarrow{1\otimes m_X} & A\otimes X \\ \downarrow{\scriptstyle \mu\otimes 1} & & \downarrow{\scriptstyle m_X} \\ A\otimes X & \xrightarrow{m_X} & X \end{array}$$

commutative where μ is the multiplication in A, and with morphisms being morphisms $f : X\to Y$ in $\mathrm{Ind}(\mathcal{C})$ making the diagrams

$$\begin{array}{ccc} A\otimes X & \xrightarrow{m_X} & X \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle f} \\ A\otimes Y & \xrightarrow{m_Y} & Y \end{array}$$

commute. We call a module of the form $A\otimes X$, for an ind-object X, with the A-action induced by multiplication, a free A-module. Note that $A-Mod$, this category we have just described, is an abelian category, since kernels and cokernels can be taken in $\mathrm{Ind}(\mathcal{C})$, and then there is a canonical

[3]One may have to move to the category of Ind-objects of $\mathcal{C}$ in order for these to make sense, in which case we will be using the tensor product of functors here defined in terms of the coend, but we gloss over this technicality for the moment.

induced A-action making them into A-modules, for instance to check kernels, let $L \rightarrowtail X$ be the kernel of $f : X \to Y$. Then, in the diagram

$$\begin{array}{ccccc} A \otimes L & \longrightarrow & A \otimes X & \xrightarrow{f} & A \otimes Y \\ \downarrow & & \downarrow{\scriptstyle m_X} & & \downarrow{\scriptstyle m_Y} \\ L & \rightarrowtail & X & \xrightarrow{f} & Y \end{array}$$

we get a uniquely induced dashed morphism by the kernel property, and universality of the kernel can be checked to ensure that this map satisfies the necessary coherence laws. By definition, for every A-module M, the action of A on M, $A \otimes M \to M$ is an epimorphism, since $1 \otimes M \to M$ is an isomorphism, and thus every A-module is a quotient of a free A-module. In this category, Etingof defines ([Eti16], 2.2) internal Homs on free modules which then extends to a definition for all A-modules (one can use the functorial free resolution from the previous procedure in order to guarantee well-definedness), in a way that are left exact so that one can define internal Ext groups, the right derived functor, which may be computed using free modules, which are acyclic for this functor.

Now, we move our attention to quadratic algebras. Given an ind-algebra A, we call A quadratic if it is of the form $T(V)/\langle R\rangle$, for $V \in \mathcal{C}$, and $R \subseteq V \otimes V$ a subobject, where we then will have $A[n]$ of the form $V^{\otimes n}/(\sum_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j})$ ([Eti16], after 3.1, and [LV12], 3.1.2). One could also, as in [LV12], define quadratic data as (V, R), $V \in \mathcal{C}$, and $R \subseteq V \otimes V$, with the morphisms as there, which is an equivalent category to that of quadratic algebras. As in [Eti16], we can define the quadratic dual of a quadratic algebra A, $A^!$ obtained from the quadratic data (V^*, R'), where R' is the largest subobject of V^* such that $R \otimes R'$ is contained in the kernel of $R \otimes V^* \otimes V^* \to V \otimes V \otimes V^* \otimes V^* \xrightarrow{(2,3)} V \otimes V^* \otimes V \otimes V^* \xrightarrow{ev \otimes ev} 1 \otimes 1 \xrightarrow{\sim} 1$. We define the Koszul complex $K^\bullet(A)$ of a quadratic algebra A to have components $K^i(A) = A \otimes (A^![i])^*$, with differential as in Etingof. We will define A to be Koszul if $K^\bullet(A)$ is exact, in the same sense as before. We have the following

Proposition 4 ([Eti16], Proposition 3.2)**.** *A quadratic algebra A is Koszul if and only if the internal Yoneda Ext algebra $\underline{Ext}^i_A(1, 1)$ is zero outside of diagonal grading i for all $i \geq 1$, where the diagonal grading comes from doing internal homs and this internal ext in the category of graded A-modules, defined similarly to before but with the usual modifications to account for grading.*

□

3. Lie Algebras and PBW

We now take a digression into the land of operads. Chapter 5 of [LV12] sets up Schur functors and S-modules in $Vect$, which readily generalizes to the case at hand by replacing the data of a $\mathbb{K}[S_n]$-module $M(n)$ by an object $M(n) \in \mathcal{C}$ (or more generally an ind-object) together with a group homomorphism $S_n \to \mathrm{Aut}(M(n))$, i.e., objects in $\mathcal{C}$ with an S_n-action on them. To set this up a bit more formally, we actually want $S_n^{op} \to \mathrm{Aut}(M_n)$, and we can define morphisms of S-modules to be collections $f_n : M(n) \to N(n)$ such that $f_n \circ \sigma = \sigma \circ f_n$ for all $\sigma \in S_n^{op}$ and all n (where σ is identified with its image in Aut of either object). The reason for the *op* is so that we could then define a Schur functor for M, $\tilde{M} : \mathcal{C} \to \mathcal{C}$[4] by $\tilde{M}(X) := \bigoplus_{n\geq 0} \int^{S_n} M(n) \otimes V^{\otimes n}$, where $V^{\otimes n}$ is given an S_n-action in the canonical way, where in this coend, we treat S_n as the one object groupoid, and the S_n-actions as functors in the canonical way- indeed this is just how we define $\otimes_{S_n}$ in this slightly generalized setting. The results of 5.1 in [LV12] generalize to this new setting with the appropriate modifications, such as replacing dimension in the Hilbert-Poincare series with quantum dimension as defined in [EGNO16], or using $\oplus_{i=1}^n 1$ to define the induced representations, and in particular, as can easily be checked, morphisms of S-modules descend to natural transformations of Schur functors, allowing us to define morphisms of Schur functors as those coming from S-module structures, forming the monoidal category we denote by $\mathrm{Schur}(\mathcal{C})$. With these definitions in hand, we can define

Definition 2 ([LV12], 5.2.1)**.** A symmetric operad for a $\mathbb{K}$-linear symmetric tensor category $\mathcal{C}$ is a monoid in the category of Schur functors on $\mathrm{Ind}(\mathcal{C})$.

Note that now we are paying attention to the fact that this whole story really has played out in $\mathrm{Ind}(\mathcal{C})$ instead of $\mathcal{C}$ which may not be able to form the direct sums that we had. Algebras over an operad can be defined precisely as in ([LV12], 5.2.3), with slight modifications for the more general case. There is a forgetful functor from operads on $\mathcal{C}$ to S-modules, and we get

Theorem 1 ([LV12], Theorem 5.4.2)**.** *The forgetful functor from operads on $\mathcal{C}$ to S-modules on $\mathcal{C}$ admits a left adjoint, the free operad, which can be constructed explicitly. To an S-module M, the free operad will be denoted $\mathcal{T}(M)$.*

□

The proof as given in the reference goes through in our more general case essentially as is.

Now, for our $\mathbb{K}$-linear symmetric tensor category $\mathcal{C}$, let X be the S-module which is zero except in grading 2, where $X(2) = 1$, the unit in $\mathcal{C}$, which we denote by 1_b, (really considered under the Yoneda embedding as an ind-object), with S_2 action $(12) \mapsto -id \in \mathrm{Aut}(1_b)$. We define the operad **Lie** as in ([Eti16], 4.1), namely, as the quotient of the operad $\mathcal{T}(X)$ by the (S-)ideal generated in degree 3 by the relation $\mathrm{im}(Id + (123) + (132)) : (1_b \otimes_{S_2} (\mathrm{Ind}_{S_1\times S_2}^{S_3} I \otimes 1_b \oplus \mathrm{Ind}_{S_2\times S_1}^{S_3} 1_b \otimes I) \to (1_b \otimes_{S_2} (\mathrm{Ind}_{S_1\times S_2}^{S_3} I \otimes 1_b \oplus \mathrm{Ind}_{S_2\times S_1}^{S_3} 1_b \otimes I))$, where $(1_b \otimes_{S_2} (\mathrm{Ind}_{S_1\times S_2}^{S_3} I \otimes 1_b \oplus \mathrm{Ind}_{S_2\times S_1}^{S_3} 1_b \otimes I)$ is precisely the degree 3 component of $\mathcal{T}(X)$, as can be seen by examining the construction.

Now, Etingof defines ([Eti16], Definition 4.1), an operadic Lie algebra in such a category $\mathcal{C}$ as an algebra for the operad **Lie** in $\mathrm{Ind}(\mathcal{C})$, and moreover a free operadic Lie algebra associated

[4] This is an abuse, in general $\tilde{M}$ takes values in ind-objects of $\mathcal{C}$.

to an ind-object V is defined as $\mathrm{FOLie}(V) = \bigoplus_{n\geq 1} FOLie_n(V) = \bigoplus_{n\geq 1}(Lie_n \otimes_{S_n} V^{\otimes n})$ (Etingof has a slightly different definition but they are equivalent, and again recall $\otimes_{S_n}$ is defined in terms of a coend). This is precisely the definition of a free **Lie**-algebra over V in the sense of [LV12], and has multiplication coming from the operadic structure of **Lie**.

We next define $E(V)$ to be the kernel of the map $FOLie(V) \to T(V)$, where $T(V)$ is the tensor algebra considered as a **Lie**-algebra by the commutator bracket, and the morphism is obtained by the universal property. With this in hand, we have

Definition 3 ([Eti16], Definition 4.6)**.** A Lie algebra is an operadic Lie Algebra L such that the natural morphism $\beta^L : \mathrm{FOLie}(L) \to L$ is zero when restricted to the idea $E(L)$ as defined above, that is to say, if β^L factors over the tensor algebra $T(L)$.

For an operadic Lie algebra L, we can define the universal enveloping algebra $U(L)$ as usual ([Eti16], 4.7) or as an adjoint to a natural functor on categories of algebras obtained from a natural transformation of operads **Lie** $\to$ **Ass** ([LV12], 5.2.14). $U(L)$ is a quotient of $T(L)$, and as usual we can get from it an associated graded $\mathrm{gr}(U(L))$, which is a commutative algebra, hence inducing a morphism of algebras $S(L) \to \mathrm{gr}(U(L))$, which allows us to define L being PBW if this is an isomorphism ([Eti16], Definition 4.10). This is a natural generalization of the classical case. Now, we see that since $L \to S(L)$ as ind-objects is monic, and L generates $S(L)$ in degree 0, if $L \to U(L)$ is not monic, since this induces $S(L) \to \mathrm{gr}(U(L))$, this cannot be monic either, so L being PBW implies $L \to U(L)$ is monic, and hence, since $\mathrm{FOLie}(L) \to L \to U(L)$ induces a unique map making the following diagram commute

$$\begin{array}{ccc} \mathrm{FOLie}(L) & \longrightarrow & U(\mathrm{FOLie}(L)) \simeq T(L) \\ \big\downarrow{\scriptstyle \beta^L} & & \big\downarrow \\ L & \longrightarrow & U(L) \end{array}$$

([Eti16], 4.7 for the isomorphism), β^L must be zero on $E(L)$, and hence L is a Lie algebra. Etingof proves the following result

Theorem 2 ([Eti16], Theorem 4.11)**.** *If an operadic Lie algebra L is a Koszul object, then L is PBW if and only if L is a Lie algebra.*

□

4. Results in the Verlinde Category

We now move on to discussing results about Lie algebras in the Verlinde category Ver_p. We assume $p \geq 5$. We start by noting as in [Ost15] that Ver_p is semisimple, with simple objects precisely L_m, for $1 \leq m \leq p-1$, where L_m is the image of $k[x]/((x-1)^m)$ as a representation of C_p. We will write 1 for L_1, the unit in Ver_p, and 1_- for L_{p-1}, since $L_{p-1} \otimes L_{p-1} = L_1$ [Ost15]. Etingof proves ([Eti16], Proposition 5.1) that L_m is a Koszul object if and only if $m = 1$ or $p-1$. We have the following

Definition 4 ([Eti16], Definition 5.3)**.** For A an algebra in $\mathcal{C}$ as before (nonnegatively graded with 0 degree component equal to 1), then we say A is an (r,s)-Koszul algebra for some $r, s \geq 1$ if $A[n] = 0$ for all $n > r$, and there is a complex $P_\bullet$ of graded projective A-modules with P_i generated by the i graded homogeneous component, and

$$H_i(P_\bullet) = \begin{cases} 1 & \text{if } i = 0 \\ A[r] \otimes P_s[s] & \text{if } i = r+s \\ 0 & \textit{otherwise}. \end{cases}$$

It was shown in [BBK02] that for $s \geq 2$, a (r,s)-Koszul algebra is quadratic. We now collect a number of results from Etingof about the PBW theorem for the Verlinde category Ver_p,

Theorem 3 ([Eti16], Theorem 6.4)**.** *If L is an operadic Lie algebra in Ver_p with $\mathrm{Hom}(L_2, L) = 0$, then L is PBW.*

□

One also derives the PBW theorem for the Verlinde Category

Theorem 4 ([Eti16], Theorem 6.6)**.** *The following are equivalent for an operadic Lie algebra L in the Verlinde category Ver_p:*

1. *L is PBW.*
2. *L is a Lie Algebra.*
3. *$\beta^L : FOLie(L) \to L$ is zero when restricted to $E_p(L)$ (the kernel of $FOLie(L)_p \to L$).*

Sketch of proof. [5] (1) $\implies$ (2). holds for any $\mathbb{K}$-linear symmetric tensor category $\mathcal{C}$, so there is nothing to check here.
(2) $\implies$ (3). This is clear, since $\beta^L|_{E(L)} = 0$ necessarily implies $\beta^L|_{E_p(L)} = 0$.
(3) $\implies$ (1). For this, one proceeds by taking a quadratic algebra $U_h(L)$ over a ring of power series in $\mathbb{K}$ called the formal universal enveloping algebra, constructed in ([Eti16], Theorem 4.11). One takes the extension groups over the exterior algebra $\wedge(L^* \oplus 1)$ in weight one higher than degree, which becomes a module over the diagonal cohomology ([PPS05], 3.1) algebra. We then use a spectral sequence, along with the Campbell-Baker-Hausdorff formula to reduce to the case of (3), that $\beta^L|_{E_p(L)} = 0$, which proves the claim (after mumbling a bit about flat deformations). □

The general case where Ver_p is replaced by an arbitrary $\mathbb{K}$-linear symmetric tensor category is currently still open, but in the same paper Etingof proves L is a Lie algebra implies L is PBW if

[5] A potentially very sketchy sketch, the equivalence of 1 and 2 will be reproved in a more elementary fashion later.

the symmetric algebra functor preserves monomorphisms.

This proof is considerably simpler. Etingof shows ([Eti16], Lemma 7.6) that a quotient of a PBW algebra is PBW, which is true in general, and then that if $\mathcal{C}$ satisfies the property of $S(-)$ preserving monics, then, for a subobject L_1 of an operadic Lie algebra L_2, if L_2 is PBW, so is L_1, since by hypothesis, $S(L_1) \to S(L_2)$ is monic, and the following diagram commutes

$$\begin{array}{ccc} S(L_1) & \longrightarrow & \mathrm{gr}(U(L_1)) \\ \downarrow & & \downarrow \\ S(L_2) & \rightarrowtail & \mathrm{gr}(U(L_2)), \end{array}$$

where the bottom arrow is monic since L_2 is PBW, which implies the top arrow is monic as well, such that L_1 is PBW. From here, one can use the fact that, for a Lie algebra L, ([Eti16], Corollary 4.9) $\mathrm{FOLie}(L)/E(L)$ is a Lie subalgebra of $T(L)$, and then since $\mathrm{FOLie}(\mathrm{FOLie}(L)/E(L)) \to \mathrm{FOLie}(L)/E(L) \hookrightarrow T(L)$ necessarily factors over $\mathrm{FOLie}(\mathrm{FOLie}(L)/E(L)) \to T(\mathrm{FOLie}(L)/E(L))$, the fact $\mathrm{FOLie}(L)/E(L)$ is a subobject of $T(L)$ proves that it is a Lie algebra. The result then follows from the fact the tensor algebra $T(L)$ is PBW, ([Eti16], Corollary 7.3).

Theorem 5 (PBW for Ver_p)**.** *An operadic Lie algebra in* Ver_p *is a Lie algebra if and only if it is PBW.*

Proof. Since Ver_p is semisimple, every monic splits, so that the symmetric algebra functor $S(-)$ preserves monics. This implies, by our discussion above, that the PBW theorem holds for Ver_p, and the claim follows. □

References

[BBK02] Sheila Brenner, Michael C. R. Butler, and Alastair King. Periodic algebras which are almost koszul. Algebras and Representation Theory, 5:331–368, 2002.

[EGNO16] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor Categories. Mathematical Surveys and Monographs. American Mathematical Society, 2016.

[EHO15] Pavel Etingof, Nate Harman, and Victor Ostrik. p-adic dimensions in symmetric tensor categories in characteristic p, 2015.

[Eti16] Pavel Etingof. Koszul duality and the pbw theorem in symmetric tensor categories in positive characteristic, 2016.

[Kra21] Henning Krause. Homological Theory of Representations. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2021.

[LV12] J.L. Loday and B. Vallette. Algebraic Operads. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.

[Ost15] Victor Ostrik. On symmetric fusion categories in positive characteristic, 2015.

[PPS05] A. Polishchuk, L. Positselski, and American Mathematical Society. Quadratic Algebras. University lecture series. American Mathematical Society, 2005.

[Rot08] J.J. Rotman. An Introduction to Homological Algebra. Universitext. Springer New York, 2008.