L-FUNCTIONS AND GEOMETRY

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ABSTRACT. In this final project, we give an expository account introducing L-functions from the point of view of number theory and geometry. The goal will be to motivate and describe how L-functions are attached to objects in algebraic geometry, how they lead to some major developments in the field, and how they relate to some open conjectures, in particular building up to the statement of the Beilenson conjectures. This should be seen as a precursor to next quarter's final project, tentatively titled "L-functions and topology," which will aim to exposit the connections between special values of L functions and homotopy theory (via K-theory, trace methods etc.).

CONTENTS

§1. INTRODUCTION

L-functions are an invariant assigned to some object of interest which output some (sometimes conjecturally) meromorphic function on the complex plane $\mathbb{C}$ which have proven to be of incredible importance over the last century in several different field of mathematics. The most famous L-function is most likely the Riemann-Zeta function $\zeta(s) = \zeta_{\mathbb{Q}}(s)$, the analytic continuation of $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$, carrying information about the prime numbers. From the perspective of Artin L-functions, this is the L-function over the rational numbers $\mathbb{Q}$ assigned to the trivial Galois representation on a one dimensional complex vector space. Similarly, from a geometric perspective, this is the L-function assigned to the terminal scheme $\mathrm{Spec}(\mathbb{Z})$, in the manner described in §3. Other applications to number theory include (but are by no means limited to) Dirichlet's famous theorem on primes in arithmetic progressions, which can be proved using analytic facts about L-functions as in [Kah20], and in one proof of the modularity theorem connecting rational elliptic curves to modular forms (more specifically newforms) of weight 2 and level $\Gamma_0(N_E)$, where N_E is the conductor of the elliptic curve E (see [DS06] discussion surrounding Theorem 8.8.3 and the sources contained therein).

In this paper, we will describe, in §2, Artin L-functions, which are L-functions attached to Galois representations. More precisely, to any number field K (or global field), one can associate an L-function $\zeta_K(s) = L_K(s, 1)$ which is defined analogously to the Riemann-zeta function using primes in the ring of integers O_K of K. The zeta functions ζ_K have values conjecturally related to the algebraic K-groups $K_*(\mathcal{O}_K)$, which is a generalization of the so-called analytic class number formula, relating (the leading term of the power series expansion at 0 of) $\zeta_K(s)$ to the group of roots of unity contained in $\mathcal{O}_K$ and the class number of $\mathcal{O}_K$. Since the torsion in $K_0(\mathcal{O}_K) = \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O}_K)$ is the class group, and the torsion in $K_1(\mathcal{O}_K) = \mathcal{O}_K^\times$ are the roots of unity in $\mathcal{O}_K$, this connection can be seen as a connection between ζ_K and torsion in K_0, K_1, which the general conjecture generalizes, which will (tentatively) be the main focus for the final project next quarter. Other connections to homotopy theory include the image of the so-called j-homomorphism in the stable homotopy groups of spheres, which Adams proved to be describable in terms of the Bernoulli numbers. Zhang took the perspective that the spectrum J corresponds to the Riemann-zeta function, and has associated "Dirichlet J-spectra" to certain Dirichlet L-functions [Zha22], though we will not speak more on this. Artin L-functions will be useful later on when giving a cohomological description of L-functions attached to schemes (and pure motives[1]).

The study of L-functions lead to some of the most important developments in mathematics in the 20th century, in particular in the foundations of modern algebraic geometry. This started with the Weil conjectures, which pertain to certain L-functions attached to varieties over a finite field $\mathbb{F}_q$. These conjectures provided some of the first applications of Grothendieck's new foundations for algebraic geometry, who proved the first three of them, in the process developing étale cohomology. This new cohomology theory, in some sense a mix of Galois cohomology and cohomology of analytic spaces, and the ideas that came along with it, were pertinent to the development of the modern theory of algebraic geometry and in turn many other fields such as number theory[2] or homotopy theory[3] As was the motivation for Grothendieck, so too will these motivate our discussion in §3 where we assign L-functions to arbitrary schemes, and later to pure motives. In the final section §4, we will aim to state the Beilenson conjectures for motivic cohomology.

Notation/Conventions

- I will write before any proof that I did as an exercise, although similar (and oftentimes probably better) proofs of these surely exist in the literature. These are simply what I came up with for the project.
- Given a global field K, and a prime $\mathfrak{p}$ in its ring of integers, we will write $k(\mathfrak{p})$ or sometimes $k_K(\mathfrak{p})$ for the residue field $\mathcal{O}_K/\mathfrak{p}$. For the most part, when discussing notation related to algebraic number theory, we will tend to follow [Sha].
- When dealing with a scheme X, we will write $X_{(0)}$ for the set of closed points of X. When R is a ring, we write $\mathrm{mSpec}(R)$ for the maximal spectrum of R, which is identified as a set with the maximal ideals in R.

[1] Attached to some suitably nice Weil cohomology theory.
[2] Where étale cohomology is widely used, as well as other p-adic cohomology theories that came along with it.
[3] Such as in the chromatic story where algebraic stacks are necessary.

We begin with some motivation, the majority of the results in this section are taken from [Sna02]. Let K be a number field. In the case of $\mathbb{Q}$, we have Riemann's zeta function $\zeta_{\mathbb{Q}}(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_{p\in\mathbb{P}}\frac{1}{(1-p^{-s})}$, where $\mathbb{P}$ denotes the set of primes. To write this more algebraically, we identify $\mathbb{P}$ with the spectrum of maximal ideals $\mathrm{mSpec}(\mathbb{Z})$, i.e., the closed points of $\mathrm{Spec}(\mathbb{Z})$. Now the expression $(p)^{-s}$ no longer quite makes sense, since (p) means the ideal $(p) \in \mathrm{mSpec}(\mathbb{Z})$, however, we can recover the integer p from $|\mathbb{Z}/(p)|$. But now we have an expression that makes sense for any ring of integers $\mathcal{O}_K \subseteq K$, so we can define:

Definition 2.1. The **Dedekind zeta function** associated to a number field K is the function $\zeta_K(s) = \prod_{\mathfrak{p}\in\mathrm{mSpec}(\mathcal{O}_K)}\frac{1}{(1-N(\mathfrak{p})^{-s})} = \sum_{I\in\mathcal{I}}(\frac{1}{N(I)^s})$, where $N(I)$ denotes the order of the quotient ring $\mathcal{O}_K/I$, and $\mathbb{I}$ is the set of nonzero ideals in $\mathcal{O}_K$.

These form holomorphic functions on the half-plane $\{\mathrm{Re}(s) > 1\}$, which can be seen from comparing with the case $K = \mathbb{Q}$: Noting that if $r = [K : \mathbb{Q}]$, there are at most r primes in $\mathcal{O}_K$ dividing any integral prime p. So, $|\zeta_K(s)| \leq \prod_{p\in\mathbb{P}}\frac{1}{|(1-p^{-s})^r|} = |\zeta_{\mathbb{Q}}(s)^r|$, and $\sum_{n\geq 1}\frac{1}{n^s}$ converges absolutely for all $s \geq 1$. One can translate this into the Dirichlet series expressions, taking the absolute value of every term, and using bounds derived from the above to get absolute convergence of the series defining $\zeta_K(s)$ in the region $s > 0$.[4]

With these in hand, one can ask about if we can "twist" the terms $N(\mathfrak{p})^{-s}$ in some way to get some larger class of L-functions with interesting properties, say, that can translate between different number fields. There is a natural way to translate invariants between number fields via examining connections to their Galois representations. So, consider a finite Galois extension E/K of number fields, and a complex representation $\rho : \mathrm{Gal}(E/K) \to \mathrm{GL}(V)$ for a finite dimensional complex vector space V.

Proposition 2.2. *These are the same as what would usually be called Galois representations, i.e., continuous homomorphisms* $\rho : G_K = \mathrm{Gal}(K^{sep}/K) \to \mathrm{GL}_n(\mathbb{C})$.

Proof. ✐ This can be proven using for instance Cartan's theorem that a closed subgroup of a Lie group is a Lie group. Thus, the image of the profinite group G_K under this continuous homomorphism must be a Lie subgroup, which would give a continuous surjection $G_K \to U$ to a compact Lie group U. We claim that this factors over some $\mathrm{Gal}(E/K)$, and to show this, it suffices to show that U is finite. We can restrict to the connected component V of the identity in U, and its inverse image H in G_K, giving a surjective map $H \to V$ from a profinite group H to a connected compact group V. Under Pontrjagin duality, this gives an injection $V^\wedge \to H^\wedge$ of (discrete) groups. Since V is connected, any $f : V \to S^1$ has connected image, so if nontrivial, so is $n \cdot f$ for all n, i.e., $V^\wedge$ is torsion-free, whereas $H = \operatorname{inj\,lim} H_i$ for finite H_i, so $H^\wedge = \operatorname{proj\,lim} H_i$ is torsion. Since $V^\wedge$ injects into $H^\wedge$, $V^\wedge = \{id\}$ is the trivial group, and then the connected component of the identity in U is a point, so U must be finite, as desired.[5] □

Remark 2.3. We emphasize that this fact above uses that the target is a Lie group, abusing the Euclidean topology[6], and in particular, the same will not be true of (continuous) ℓ-adic representations $G_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ later on, which will be the correct notion for our definitions.

[4] Alternatively one can work directly with the Dirichlet series and use bounds arrived at by studying the geometry of embeddings as in [Sha].

[5] There should also be an analytic proof of this fact without appealing to Pontrjagin duality by utilizing a similar proof as in a problem on the first homework to classify compact subgroups of S^1, but this proof is the shortest and cleanest that I can come up with.

[6] One can also give the target the discrete topology and the same proposition is true, but even easier to show.

Before define the Artin L-function associated to a complex Galois representation ρ, we have to introduce some notation. Given a prime $\mathfrak{p}$ in $\mathcal{O}_K$, we fix a prime $\wp$ in $\mathcal{O}_{K^{sep}}$[7], define $G_\wp$ as the subgroup $G_\wp \subseteq G_K$ of all elements φ with $\varphi(\wp) = \wp$. This subgroup is identified with the absolute Galois group of the local field at the place $\mathfrak{p}$, and has inertia subgroup $I_\wp$, arising as the subgroup of the map $G_\wp \longrightarrow G_{k(\mathfrak{p})}$, and we have that $G_\wp / I_\wp \simeq G_{\mathcal{O}_K/\mathfrak{p}}$ is topologically generated by the Frobenius element $\mathrm{Frob}_\mathfrak{p}$. Then, $\mathrm{Frob}_\mathfrak{p}$ acts on the fixed point subspace $V^{G_\wp}$, allowing us to define:

Definition 2.4. If $\rho : G_K \to \mathrm{GL}(V)$ is a complex Galois representation, the **Artin L-function** $L_K(s,\rho)$ is defined by:

$$L_K(s,\rho) = \prod_{\mathfrak{p} \in \mathrm{mSpec}(\mathcal{O}_K)} (\det(1 - N(\mathfrak{p})^{-s} \, \mathrm{Frob}_\mathfrak{p} |_{V^{G_\wp}})^{-1}).$$

This is independent of the choice of $\wp$ over $\mathfrak{p}$ since any two choices are conjugate and the determinant is invariant under conjugation. We have:

Proposition 2.5 ([Sna02] Proposition 1.1.3)**.** *The Artin L-functions have the following properties*[8]*:*
(i) $L_K(s,\rho)$ is a holomorphic function on the half-plane $\mathrm{Re}(s) > 1$, and admits a meromorphic extension to the whole plane.
(ii) We have $L_K(s, \rho_1 \oplus \rho_2) = L_K(s,\rho_1) L_K(s,\rho_2)$.
(iii) If F/K is an arbitrary extension of number fields, and ρ is a Galois representation for F, then:

$$L_K(s, \mathrm{Ind}_{G_F}^{G_K} \rho) = L_F(s,\rho).$$

Sketch. (i) We sketch the first half of the claim. The action of $G_{k(\mathfrak{p})}$ on $V^{G_\wp}$ factors over a finite cyclic group, so the action can be diagonalized, and we can write, up to conjugating $\rho(\mathrm{Frob}_\mathfrak{p}) = diag(\zeta_1, \ldots, \zeta_{\dim V^{G_\wp}})$, where the ζ_i are roots of unity. In particular, $\det(1 - N(\mathfrak{p})^{-s} \, \mathrm{Frob}_\mathfrak{p} |_{V^{G_\wp}}) = \prod (1 - N(\mathfrak{p})^{-s} \zeta_i)$. Taking the associated Dirichlet series expansion, taking the absolute value of every term, this gives us a bound on the coefficients in terms of those of $\zeta_K(s)^{\dim(V)}$, and this allows us to get absolute convergence of $L_K(s,\rho)$ when $\mathrm{Re}(s) > 1$. For the second half of the claim, the usual proof proceeds by using (ii) and (iii) together with a result of Brauer to reduce to the case when V is 1-dimensional and ρ factors over a cyclic quotient of G_K.

(ii) Is clear.

(iii) For (iii), we take inspiration from Mackey, who proved a decomposition formula [Con] for $\mathrm{Res}_H^G \mathrm{Ind}_K^G \rho$ when H, K are two subgroups of a (finite) group G (applied using some $G = \mathrm{Gal}(E/R)$ which ρ factors over). For concreteness, we restrict to the case when V is 1-dimensional and $\mathrm{Gal}(E/F)$ is cyclic of order n (the general case, combined with (ii) and Brauer's theorem follows from these cases for an alternative method). Mackey's formula tells us that we can express $\mathrm{Res}^{\mathrm{Gal}(E/K)}_{\mathrm{Gal}(E_\wp/K_\mathfrak{p})} \mathrm{Ind}^{\mathrm{Gal}(E/K)}_{\mathrm{Gal}(E/F)}(\rho)$ (abusively identifying $\wp$ with its restriction to E) in terms of a sum of induced representations $\mathrm{Ind}^{\mathrm{Gal}(E_\wp/K_\mathfrak{p})}_{\sigma_i \mathrm{Gal}(E_{\mathcal{q}_i}/F_{\mathfrak{q}_i}) \sigma_i^{-1}}(c^\sigma \rho)$, one for each prime $\mathfrak{q}_i$ in

[7] This prime will be infinitely generated, and this choice amounts to a coherent choice of primes over $\mathfrak{p}$ for some cofinal system of Galois extensions of K, recovering the same notions used in the definitions of Snaith [Sna02], but we prefer this definition since it's more closely connected to what will be done in the next section.

[8] Snaith's property (ii) does not apply in our situation since we defined our Artin L-functions directly from representations of the absolute Galois group, it just says that the definition is independent of the choice of E/K which ρ factors over the Galois group of.

F lying over $\mathfrak{p}$. If the Frobenius of $\mathfrak{q}_i$ acts as ζ on V, then the induced representation over $\mathfrak{q}_i$, after taking invariants for the inertia subgroup, has $\mathrm{Frob}_\mathfrak{p}$ acting on it as $diag(\zeta, \zeta_m\zeta, \dots, \zeta_m^{m-1})$, where $[k_F(\mathcal{q}_i) : k_K(\mathfrak{p})] = m$ (essentially, on these fixed point parts, we are inducing from the cyclic group $\mathbb{Z}/r\mathbb{Z}$ up to $\mathbb{Z}/m\mathbb{Z}$). Thus, the determinant of $1 - N(\mathfrak{p})^{-s}\,\mathrm{Frob}_\mathfrak{p}$ on this subspace is $\prod_{i=0}^{m-1}(1 - N(\mathfrak{p})^{-s}\zeta\zeta_m^i) = (1 - N(\mathfrak{p})^{-ms}\zeta)$. Using that $N(\mathfrak{p})^m = N(\mathfrak{q}_i)$, we get that the factors in the product in Definition 2.4 defined for the primes lying over $\mathfrak{p}$ in the exression for the L-function over F multiply out to give exactly the expression for the multiplicative term attached to the prime $\mathfrak{p}$ in the L-function for K on the induced representation, giving the desired equality. □

We conclude this section with an advertisement for next quarter. One of the reasons Artin L-functions, and in particular Dedekind zeta functions, are studied, is for their conjectural connections to algebraic K-theory. The first result in this direction is the following:

Theorem 2.6. *Let K be a number field, r_1 the number of distinct real embeddings and $2r_2$ the distinct number of complex embeddings. Then, we have:*

$$\lim_{s\to 0} s^{1-r_1-r_2}\zeta_K(s) = -\frac{|\,\mathrm{Tors}(K_0(\mathcal{O}_K))|R_1(K)}{|\,\mathrm{Tors}(K_1(\mathcal{O}_K))|},$$

where the denominator is the order of the group of roots of unity in $\mathcal{O}_K$ and the numerator is the class number times $R_1(K)$, the regulator, which is defined in terms of the covolume of the image of the logarithm map defined for instance in [Sha].

This gives a connection between these zeta functions and algebraic K-theory, which we don't understand that well on rings of integers.[9] What we will tentatively describe next quarter is the proof of the above theorem, together with its extension to the so-called Lichtenbaum conjecture (taken in this form from [Sna02]):

Conjecture 2.1 (Lichtenbaum)**.** *For integers $m \geq 2$, we have*

$$\zeta_K(1-m)^* = \pm 2^n \frac{|K_{2m-2}(\mathcal{O}_K)|R_m(K)}{\mathrm{Tors}(K_{2m-1}(\mathcal{O}_K))},$$

where $n \geq 0$ is an integer.

We will leave the actual definitions of $\zeta(1-m)^*$ and $R_m(K)$ until next time.

[9]This is hyperbole, while we don't know the complete computations, and plenty of parts are still conjectural, we have a pretty good idea of what these look like in many cases. $K(\mathbb{Z})$ is only known up to a number theory conjecture, but we do know what its p-completion looks like at primes which are not counterexamples to Vandyver's conjecture, see Weibel [Wei05] for an exposition.

§3. L-Functions in Algebraic Geometry

In 1949, André Weil proposed some conjectures related to smooth projective varieties over a finite field [Wei49]. He defined:

Definition 3.1. Let X be a smooth projective variety over the finite field $\mathbb{F}_q$. Define the **zeta function** of X to be:

$$Z(X,t) = \exp\Big(\sum_{m\geq 1} |X(\mathbb{F}_{q^m})|\frac{t^m}{m}\Big) \in \mathbb{Q}[[t]].$$

This is defined in such a way that the logarithmic derivative of this power series gives the generating function for the sequence $a_m = |X(\mathbb{F}_{q^m})|$. Weil conjectured that if X is smooth projective of dimension d, then we have, paraphrasing from Milne [Mil13]:

Conjecture 3.1 (Weil [Wei49], (1)-(3) theorems of Grothendieck, (4) theorem of Deligne)**.** *(1) $Z(t)$ is a rational function, which can be written as $Z(t) = \frac{P_1(t)\dots P_{2d-1}(t)}{P_0(t)\dots P_{2d}(t)}$, where $P_0(t) = 1 - s$, $P_{2d}(t) = 1 - q^d t$, and for the other r, $P_r(t) = \prod_{i=1}^{\beta_r}(1-\alpha_{r,i}t)$.*
(2) The β_r act as we would expect Betti numbers to from complex geometry.
(3) The $Z(s)$ satisfy a functional equation $Z(\frac{1}{q^d t}) = \pm q^{d\chi/2} t^{\chi} Z(t)$ with $\chi = \sum_r (-1)^r \beta_r$.
(4) (Riemann hypothesis) The $\alpha_{r,i}$ are algebraic integers with $|\alpha_{r,i}| = q^{r/2}$.

For a sketch of Grothendieck's proofs of the first 3, we refer to [Mil13]. They key insight Grothendieck had was that one could define so-called étale cohomology, which, taken with coefficients in $\mathbb{Q}_\ell$[10] for a prime $\ell \nmid q$, satisfies a suitably analogue of the Lefschetz fixed point theorem from differential topology. Since the fixed points of the Frobenius $x \mapsto x^{q^m}$ are exactly the $\mathbb{F}_{q^m}$-points of X, it should at least be intuitively clear why the above zeta functions should be connected to actions of Frobenii on étale cohomology.

One may wish to define similar zeta functions for other schemes. A natural situation to consider is when everything is finite at every prime, so we can get meaningful counts, explicitly:

Definition 3.2. Let $X/\mathrm{Spec}(\mathbb{Z})$ a finite type scheme, and let $X_{(0)}$ be the set of dimension 0 (i.e. closed) points of X. We define the **zeta function** of X as:

$$\zeta(X,s) = \prod_{x\in X_{(0)}} \frac{1}{1 - |k(x_0)|^{-s}}.$$

This converges for $\mathrm{Re}(s) > \dim(X)$, and has an analytic continuation to a meromorphic function on the whole plane.

Example 3.3. (1) If $X = \mathrm{Spec}(\mathbb{Z})$, then $\zeta(X,s) = \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}}$ is Riemann's zeta function. Similarly, if $X = \mathrm{Spec}(\mathcal{O}_K)$ for a number field K, then $\zeta(X,s) = \zeta_K(s)$.
(2) If X is a smooth projective scheme over a finite field $\mathbb{F}_q$, it can be shown as in [Mil13] that $\zeta(X,s) = Z(X, q^{-s})$. For a concrete example, take the degenerate case $X = \mathrm{Spec}\,\mathbb{F}_q$, then $Z(X,t) = \exp\Big(\sum_{m\geq 1}\frac{t^m}{m}\Big) = \exp(-\log(1-t)) = \exp\Big(\log\Big(\frac{1}{1-t}\Big)\Big) = \frac{1}{1-t}$, $\zeta(X,s) = \frac{1}{1-q^{-s}}$, and we can clearly see the desired equality.
(3) For a slightly less trivial example, let's consider the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$. The closed points correspond to places for the global field $\mathbb{F}_q(t)$. Aside from the point at ∞ with residue field $\mathbb{F}_q$, these points are identified with prime ideals in $\mathbb{F}_q[t]$, in turn identified

[10] This is meant to carry a topology, so one has to be slightly careful with the definition of ℓ-adic étale cohomology.

with monic irreducible polynomials $f(t)$ over $\mathbb{F}_q$, with residue field of order $q^{\deg(f)}$ (since $[\mathbb{F}_q[t]/f(t) : \mathbb{F}_q] = \deg(f)$). Thus, the Euler product expansion is

$$\zeta(\mathbb{P}^1_{\mathbb{F}_q}, s) = \frac{1}{1-q^{-s}} \prod_{f \in \mathbb{F}_q[t] \text{ monic irreducible}} \frac{1}{1-q^{-s\deg(f)}}.$$

Expanding out this Euler product and using that each nonzero ideal in $\mathbb{F}_q[t]$ is determined uniquely by a monic polynomial, we get

$$\zeta(\mathbb{P}^1_{\mathbb{F}_q}, s) = \frac{1}{1-q^{-s}} \sum_{f \in \mathbb{F}_q[t] \text{ monic}} \frac{1}{q^{-s\deg(f)}}.$$

As there are q^m monic polynomials of degree m over $\mathbb{F}_q$, we can rewrite this as:

$$\zeta(\mathbb{P}^1_{\mathbb{F}_q}, s) = \frac{1}{1-q^{-s}} \sum_{m \geq 1} \frac{q^m}{q^{-sm}} = \frac{1}{1-q^{-s}} \sum_{m \geq 1} q^{m(1-s)}.$$

We have a factor $\frac{1}{1-q^{-s}}$ coming from the point at ∞, which is the zeta function associated to a point, and the other factor is the same as the zeta function associated to the affine line.

This last example demonstrates the following lemma, which follows straight from Definition 3.2:

Lemma 3.4. *Let $X/\mathrm{Spec}(\mathbb{Z})$ be a finite type scheme, and U, V two subschemes of X with $X_{(0)} = U_{(0)} \coprod Z_{(0)}$. Then $\zeta(X,s) = \zeta(U,s)\zeta(V,s)$. The same holds for any countable decomposition of X.*

Example 3.5. Let's examine the last computation over $\mathbb{Z}$ as well: So, let $X = \mathbb{A}^1 = \mathrm{Spec}\,\mathbb{Z}[t]$. We have a infinite decomposition $X = \coprod_{p \in \mathbb{P}} \mathbb{A}^1_{\mathbb{F}_p}$. Rewriting our expression from Example 3.3 as $\zeta(\mathbb{A}^1_{\mathbb{F}_p}, s) = \sum_{m \geq 1} p^{m(1-s)} = \frac{1}{1-p^{1-s}}$, and using this decomposition, we get the zeta function

$$\zeta(X,s) = \prod_{p \in \mathbb{P}} \zeta(\mathbb{A}^1_{\mathbb{F}_p}, s) = \prod_{p \in \mathbb{P}} \frac{1}{1-p^{1-s}} = \zeta(\mathrm{Spec}\,\mathbb{Z}, s-1).$$

A similar formula holds for Dedekind zeta functions over other rings of integers.

I figured I would just write up some examples for fun but this last example seems like it shouldn't be hard to generalize, and it looks like, up to a degree shift, the $\mathbb{A}^1$-invariance that shows up in motivic homotopy theory (which is a reflection of the cohomological interpretation).

Theorem 3.6 (, "$\mathbb{A}^1$-invariance" of zeta functions for $\mathbb{Z}$-schemes)**.** *Let X be a finite type scheme over $\mathbb{Z}$. Comparing ζ functions, we have that $\zeta(X \times \mathbb{A}^1, s) = \zeta(X, s-1)$.*

Proof. By Lemma 3.4 and induction on dimension, we may assume without loss of generality that X is affine, say $X = \mathrm{Spec}\,R$. A closed point of $X \times \mathbb{A}^1$ corresponds to a maximal ideal of $R[t]$, and since this was assumed finite type over $\mathbb{Z}$, the residue field at this point is a finite field $\mathbb{F}_q$, for some q. The only subrings of a finite field are finite fields, to the prime ideal $\mathfrak{m}$ of R that this pulls back to is also maximal, and the correspondence theorem gives us a correspondence between maximal ideals of $R[t]$ containing $\mathfrak{m}$ and maximal ideals of $R/\mathfrak{m}[t]$. That is to say, we have $X = \coprod_{\mathfrak{m} \in \mathrm{mSpec}(R)} \mathbb{A}^1_{k(\mathfrak{m})}$. We then have:

$$\zeta(X \times \mathbb{A}^1, s) = \prod_{x \in X_{(0)}} \zeta(\mathbb{A}^1_{k(x)}, s) = \prod_{x \in X_{(0)}} \zeta(k(x), s-1) = \zeta(X, s-1),$$

from the case of a finite field $k(x)$ covered in Example 3.3 (3). □

Corollary 3.7. *If $X = \mathbb{P}^n_{\mathbb{F}_q}$ is n-dimensional projective space over a finite field, then $\zeta(X,s) = \prod_{i=0}^{n} \frac{1}{1-q^{s-i}}$. If $X = \mathbb{P}^n$ is n-dimensional projective space over $\mathbb{Z}$, then $\zeta(X,s) = \zeta(s)\ldots\zeta(s-n)$, where $\zeta(s)$ is Riemann's zeta function.*

This shows that the requirement $\mathrm{Re}(s) > \dim(X)$ for convergence is sharp. Theorem 3.6 also allows us to prove this bound:

Proposition 3.8. *Let X be a finite type $\mathbb{Z}$-scheme. Then $\zeta(X,s)$ converges for* $\mathrm{Re}(s) > \dim X$.

Proof. First note that Lemma 3.4 allows us to reduce to the case when X is connected, and since we care about the closed points, we may assume without loss of generality that X is reduced. Since X is of finite type over $\mathbb{Z}$, it is Noetherian, and thus has finitely many irreducible components, so another application of Lemma 3.4 allows us to assume that X is connected, reduced, and irreducible, hence integral. By taking an affine open U, which necessarily contains the generic point (the only dimension n point), and using induction on dimension together to get the result for $X\backslash U$, together with 3.4, we may assume that X is an affine integral scheme. We induct on the relative dimension of $X/\mathbb{Z}$, the case of the relative dimension 0 means either X is of dimension zero and $X = \operatorname{Spec} \mathbb{F}_q$ for a finite field $\mathbb{F}_q$, in which case the result follows by 3.3 (2), or else X is of dimension 1. If X is of dimension 1, then its generic fiber is a dimension zero connected, reduced, scheme over $\mathbb{Q}$, so a finite field extension K. Then X[11] is a localization of $\operatorname{Spec} \mathcal{O}_K$ at finitely many primes, and the result follows from the case of the Dedekind zeta functions discussed in §2 (after cancelling out the terms coming from the finitely many primes localized out). Now, if X has relative dimension n, if the map $X \to \operatorname{Spec}\mathbb{Z}$ has image a single closed point, then X is a scheme over $\mathbb{F}_q$ for some q, and we can find an affine open subscheme U of X with a finite degree map to $\mathbb{A}^n_{\mathbb{F}_{q^m}}$ for some $m \geq 1$, whence by looking at the fibers over any point and then noting the zeta function for X divides $\zeta(\mathbb{A}^n_{\mathbb{F}_q}, s)^{\deg(X \to \mathbb{A}^n_{\mathbb{F}_q})}$, the result follows from the case of $\mathbb{A}^n_{\mathbb{F}_{q^m}}$ and the result for $X\backslash U$ which holds by induction. Otherwise, the image of $X \to \operatorname{Spec}\mathbb{Z}$ is open, in which case, the rationalization of X is a scheme over a number field K, and up to localizing at finitely many primes (which we may again do by induction on dimension), X is a scheme over $\operatorname{Spec}\mathcal{O}_K$ for an $a \in \mathcal{O}_K$. We can again find an open subscheme U of X which has a finite degree map onto affine n-space $\mathbb{A}^n_{\mathcal{O}_K}$, and then we win from Theorem 3.6 and induction on dimension. More or less what we have just done is look for representatives of each birational equivalence class that we understand and using induction on dimension. □

If one is doing geometry, we like to work over a field, so one may want to ask how to assign a zeta function to a proper smooth scheme over $\mathbb{Q}$, or more generally a number field K (or a local field, but more on this later). One option to do this is the Hasse-Weil zeta function. If $X/\operatorname{Spec} K$ is a smooth projective scheme over a number field K, we say that X has **good reduction** at a finite place $\mathfrak{p}$ if there is a smooth projective scheme $\mathcal{X}$ over $\operatorname{Spec}((\mathcal{O}_K)_{(\mathfrak{p})})$ which restricts to X upon taking the generic fiber (pulling back along $\operatorname{Spec} K \to \operatorname{Spec}((\mathcal{O}_K)_{(\mathfrak{p})})$), such a scheme is called a smooth projective model of X. In this case, one can define:

Definition 3.9 ([Kah20])**.** If $X/\operatorname{Spec} K$ is a smooth projective scheme with good reduction at a place $\mathfrak{p}$, we define the **local zeta factor** of X at $\mathfrak{p}$ by:

$$L_{\mathfrak{p}}(X,s) = \zeta(\mathcal{X}_{sp}, s),$$

[11]Or an open subset at least after throwing away some finite set of primes, which can be done since the absolute dimension 0 case was already handled.

where $\mathcal{X}$ is a smooth projective model of X over $\mathrm{Spec}((\mathcal{O}_K)_{(\mathfrak{p})})$, and $\mathcal{X}_{sp}$ is the special fiber of $\mathcal{X}$, a smooth projective scheme over the finite field $\mathcal{O}_L/\mathfrak{p}$, and the zeta function on the right is as in Definition 3.2. If S is some set of finite places such that X has good reduction on all of the primes not contained in S, we define the **Hasse-Weil zeta function** of X to be:

$$\zeta_S(X,s) = \prod_{\mathfrak{p}\in \mathrm{mSpec}(\mathcal{O}_K)\setminus S} \zeta_{\mathfrak{p}}(X,s).$$

The local L-factors do not depend on a choice of projective model, as we shall soon se. Since a scheme X as above has good reduction at all but finitely many primes, the Hasse-Weil zeta function can be defined away from the bad primes, so that we miss only finitely many local factors. To Serre, this was not entirely satisfactory, he wanted a zeta function that would work for all smooth projective schemes over a number field. Define the **Local L-factor of weight i** $L_{\mathfrak{p}}(X,i,s) = \det\left(1 - N(\mathfrak{p})^{-s}\pi_{\mathfrak{p}}|_{H^i_{\acute{e}t}(X,\mathbb{Q}_\ell)^{I_{\mathfrak{p}}}}\right)^{-1}$, where $\pi_{\mathfrak{p}}$ denotes the local Frobenius acting on the inertia fixed points of $H^i_{\acute{e}t}(X,\mathbb{Q}_\ell)$ via the induced (continuous) Galois action of G_K on the étale cohomology, for a prime ℓ not dividing $N(\mathfrak{p})$. We will use the following, which arises from base change theorems in étale cohomology and Grothendieck's solution to the Weil conjectures:

Lemma 3.10 ([Kah20]). *If X has good reduction at a place $\mathfrak{p}$, and ℓ is a prime not dividing $N(\mathfrak{p})$, we have that $L_{\mathfrak{p}}(X,s)$ is determined by the ℓ-adic étale cohomology. More precisely, we have $P_i(\bar{X}, N(\mathfrak{p})^{-s})^{-1} = L_{\mathfrak{p}}(X,i,s)$ where $\bar{X}$ is the special fiber of a smooth proper model of X over $\mathcal{O}_K$, and $P_i(\bar{X},t)$ is the polynomial of the synonymous name from the statement of the Weil conjectures. In this case, the Galois action of G_K on the ℓ-adic étale cohomology is unramified at $\mathfrak{p}$.*

Recall that the local zeta factor for X is $\prod_{i=0}^{2\dim(X)} P_i(\bar{X}, N(\mathfrak{p})^{-s})^{-1}$. The lemma above (together with facts about ℓ-adic étale cohomology) simultaneously shows that the local L-factors for fixed i, $L_{\mathfrak{p}}(X,i,s)$ and the P_i are independent of the choice of smooth projective model $\mathcal{X}$ of X and the prime ℓ. This definition of the local L-factor did not use that X had good reduction, and depended only on X, which lead Serre to make the conjecture (well, really several conjectures were made in this paper)

Conjecture 3.2 (Serre, [Ser69]). *The local L-factor of weight i does not depend on the choice of a prime ℓ not dividing $N(\mathfrak{p})$.*

One can define a zeta-function associated to X (which is conjecturally dependent on a choice of ℓ) $\zeta(X,s)$ by fixing a prime ℓ with good reduction, taking S to be the set of primes with bad reduction, and setting $\zeta(X,s)_\ell = \zeta_S(X,s)\cdot\prod_{\mathfrak{p}\in S} L_{\mathfrak{p}}(X,i,s)_\ell$,[12] where the subscript makes it explicit we have made a choice of ℓ.[13] There are some known cases when this does not depend on the choice of i, some explicit ones are when $i = 0, 1$, $i = 0$ this is just the case of points, since the étale cohomology is just detecting (geometrically) connected components, and for $i = 1$, this is a theorem of Grothendieck, using that the ℓ-adic cohomology can be described as a Tate module of the picard scheme (see the discussion in [Kah20] 5.6 for more details).

[12] One can also define factors for the Archimedean places using Hodge theory to examine the analytification of our schemes as complex manifolds. As a complete not forced at all remark, hey this connects to the Wirtenger derivatives we had on the homework since ∂ and $\bar{p}$, acting on the complexified cotangent bundle of a complex manifold, are exactly what gives rise to our Hodge structure!

[13] Though many authors will not include such a subscript, either by assuming Serre's conjecture or by keeping ℓ implicit.

§4. The Beilenson Conjectures

In this section, we will attempt to state the Beilenson conjectures as discussed in [DS91]. Originally, I had planned to discuss L-functions attached to pure motives in the last section, but due to time constraints and (moreso) the fact that this project is not supposed to go over 10 pages[14], we will [Kah20] for a thorough reference, and just say a few words about them.

The motivation for defining L-functions attached to motives is to unify the case of algebraic geometry and of Artin L-functions. The definition comes from the local L-factors $L_{\surd}(X, i, s)$ defined in the previous section, where one can take a product over all primes to define $L(X, i, s)$ (which is conjecturally independent of a choice of ℓ). What one typically does, is instead of working with Chow motives, one replaces the correspondences coming from Chow cohomology (cycles up to rational equivalence) in the definition with some other suitably nice equivalence relation (e.g. associated to a Weil cohomology theory), to define a category of pure motives, which depends on an equivalence relation $\sim$. There are some specific subcategories, such as the category of abelian varieties over K up to isogeny, and the category of Artin motives, which do not depend on $\sim$. The category of Artin motives is equivalent to the category of continuous Galois representations with coefficients in a (chosen) base field (which the defined category of motives depends on), and the associated L-functions are precisely the Artin L-functions from §2. The local L-factor attached to the motive represented by a smooth projective variety X at a prime of good reduction is just the local L-factor as described before, so this framework is supposed to unify the L-functions appearing in §2 with those in §3.

In an effort to not go over 10 pages, we will follow [DS91] for stating the Beilenson conjectures, and we will attempt to keep the discussion brief. We will only consider the case of a smooth projective scheme so we don't have to worry about defining (what I think would be called) log-Deligne cohomology. Fix a proper smooth variety X over $\mathbb{Q}$. We define $H^p_{\mathcal{M}}(X, \mathbb{Q}(q))$ to be motivic cohomology (defined for instance via Bloch's higher chow groups) with the usual grading convention. Deninger-Scholl define $H^p_{\mathcal{M}}(X, \mathbb{Q}(q))_{\mathbb{Z}}$ to be $H^p_{\mathcal{M}}(X, \mathbb{Q}(q))$ when $q > p$, and $\mathrm{Im}(H^p_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(q)) \to H^p_{\mathcal{M}}(X, \mathbb{Q}(q)))$ for $q \leq p$, for a proper regular model $\mathcal{X}$ of X over $\mathbb{Z}$[15]. One can define Deligne cohomology by an explicit chain complex of sheaves on a complex manifold, and get a short exact sequence $0 \to F^q H^p_{dR}(X) \to H^p_B(X, \mathbb{R}(q-1)) \to H^{p+1}_{\mathcal{D}}(X, \mathbb{R}(q)) \to 0$[16], so by the usual fact on how det behaves with respect to short exact sequences of vector spaces, they define $B_{p,q} = \det H^p_B(X, \mathbb{Q}(q-1)) \otimes F^q H^p_{dR}(X)^\wedge$ defines a rational vector space with a canonical injection to $\det H^{p+1}_{\mathcal{D}}(X, \mathbb{R}(q))$. Using a cosimplicial resolution of X, $\mathbb{A}^1$-invariance, and some resulting spectral sequences, [DS91] construct the **regulator map** $r_{\mathcal{D}} : H^p_{\mathcal{M}}(X, \mathbb{Q}(q))_{\mathbb{Z}} \to H^p_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(q))$. We end by stating Beilinson's conjectures (assuming that L-functions attached to motives work sufficiently nicely):

Conjecture 4.1 (Beilenson [Bei85], [DS91] 3.1)**.** *For $q > \frac{p+1}{2}$:*
(1) $r_{\mathcal{D}}$ is an isomorphism after tensoring the source with $\mathbb{R}$.
(2) $r_{\mathcal{D}}(\det H^p(X, \mathbb{Q}(q))_{\mathbb{Z}}) = B_{p-1,q} L(X, p-1, p-q)^*$, where $L(X, p-1, p-q)^*$ means the leading coefficient of the power series expansion of this L-function at p-q.

[14] It may be over by a little bit due to the bibliography.

[15] From consulting the literature does not seem to be known that one always exists in general, and some sources seem to define this using a proper flat model, but then conjecture that it's independent of a choice of proper flat model.

[16] B is for Betti cohomology, and the filtration is the Hodge filtration, which can be defined for schemes over any base ring.

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