UCLA Algebraic Topology Participating Seminar F23:
The Adams Spectral Sequence

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The goal of this learning seminar is to build up the necessary machinery to define the Adams spectral sequence and present some computations with a particular focus on the stable homotopy groups of spheres. The seminar is structured to have not assume familiarity with spectral sequences so as to be inclusive of more junior students. This will also serve as preparation for a winter seminar on chromatic homotopy theory (finite generation of homotopy groups of spheres).

Week 1 Introduction to spectral sequences

Loosely following the introduction to [McC01], we will motivate the study of spectral sequences by using them as black–boxes to recover some familiar theorems (Mayer-Vietoris, Universal Coefficient Theorem, etc.) and see how they may be used to reason about higher homotopy groups.

Week 2 Basic computations with the Serre and Atiyah-Hirzebruch spectral sequences

This talk will do some basic spectral sequence computations to see how these are used in practice. Possible computations could be things along the lines of $\pi_4(S^2)$, $\pi_{2n-1}(S^n) = \mathbb{Z}$ for n even, computing the cohomology of $\mathbb{C}P^n$, proving the universal coefficient theorem, etc. A source for many examples is [McC01].

Week 3 Building spectral sequences from exact couples and filtered graded modules

In this talk, we will get under the hood of spectral sequences and show how they may be constructed from certain diagrams called *exact couples* or from filtered chain complexes. We will also discuss convergence properties of spectral sequences. A good reference for these topics is [McC01, Chapters 2 and 3].

Week 4 Triangulated categories, derived functors, and the Grothendieck spectral sequence

Here we will introduce triangulated/stable categories through the lens of chain complexes. In particular, this talk will define fiber and cofiber sequences, as well as derived functors, and how to use injective resolutions to compute them, with examples.

Week 5 (Filtered) Spectra

What are spectra? How do they connect to the derived categories of last week? What is a filtered object in spectra, or more generally a stable category, and how do these give rise to spectral sequences? This talk will aim to answer these questions, providing a lightning introduction to some of the main players in modern stable homotopy theory.

Week 6 Epimorphism classes and $\mathcal{E}$–injective resolutions

In this talk, we will discuss the theory of epimorphism classes and injective resolutions from [PP21], a number of examples, and (time permitting) the equivalence with adapted homology theories.

Week 7 Miller's Adams spectral sequence, Adams towers, and the Adams spectral sequence.

Continuing from week 6, we discuss how the epimorphism classes of [PP21] give rise to spectral sequences by taking injective resolutions, and in particular, how to obtain the original Adams spectral sequence from this framework.

Week 8 Easy computations with the Adams spectral sequence

In this talk, we will compute *ko*–cohomology using the Adams spectral sequence, where you can find Adams resolutions by hand. To be expanded later.

Week 9 One big computation with the Adams spectral sequence

In this talk, we will use the Adams spectral sequence to compute some homotopy groups of spheres. To be expanded later.

Week 10 Synthetic spectra

In this talk, we will discuss how the epimorphism classes of [PP21] give rise to "deformations" of a stable category, and how this connects the homotopy theory (the "generic fiber") to the algebra (the "special fiber") in analogy with deformations of p-adic formal schemes.

References

[McC01] J. McCleary. *A User's Guide to Spectral Sequences.* A User's Guide to Spectral Sequences. Cambridge University Press, 2001.

[PP21] Irakli Patchkoria and Piotr Pstragowski. Adams spectral sequences and franke's algebraicity conjecture. *arXiv preprint arXiv:2110.03669*, 2021.