ON GENERALIZED FOURIER SERIES

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1. Introduction

In this note, we aim to answer the following (initially vague) question:

Question 1.1. *If we have a complex Banach space B with a nice enough action of the circle S^1, can we define some notion of Fourier series in B such that every vector $v \in B$ can be given a Fourier expansion.*

Let's start by figuring out a precise form of the question to ask such that it has a chance of being true. First, as topologies are involved everywhere, it seems natural to ask that the action of the circle be continuous. Furthermore, since we are working with a Banach space B, we have a norm $\|\bullet\|_B$ on it, and our group action should preserve this norm- that is, the group should act through isometries. Since we will repeatedly use these kinds of spaces throughout, let's give them a name:

Definition 1. A complex Banach space B equipped with a continuous action of the circle through isometries will be termed an *S^1-Banach space*.

These turn out to be all the conditions we have to impose, but now we have to ask about the Fourier coefficients themselves- what is a Fourier coefficient? Classically, we may be dealing with some function space, such as the space $\ell^2(S^1, \mathbb{C})$ of ℓ^2 complex-valued functions on the circle, or the space of continuous functions on the circle $C(S^1, \mathbb{C})$ with the supremum norm. In each of these cases, the Fourier coefficients are scalar multiples of functions of the form $e^{ix} \mapsto e^{inx}$ for $e^{ix} \in S^1$ and some $n \in \mathbb{Z}$. The distinguishing property of these functions is that given an element $e^{iz} \in S^1$, e^{iz} acts on $e^{ix} \mapsto e^{inx}$ by taking it to $e^{ix} \mapsto e^{in(x+z)} = e^{inz} \cdot e^{inx}$. That is, if $e^{iz}\{f\}$ denotes the action of e^{iz}, then the space of all f such that $e^{iz}\{f\} = e^{inz} \cdot f$ for all $e^{iz} \in S^1$ identifies with the "space of nth Fourier coefficients" for these circle actions. This motivates the following definition:

Definition 2. If B is an S^1-Banach space, the *space of nth Fourier coefficients* $B[n]$ of B is the subspace of vectors v such that for all $e^{iz} \in S^1$, $e^{iz}\{v\} = e^{inz} \cdot v$.

Definition 3. There are projection maps $p_n : B \to B[n]$ defined as the limit

$$p_n(v) = \lim_{m \to \infty} \frac{1}{m!} \sum_{k=1}^{m!} e^{-2\pi i k n/m!} \cdot (e^{2\pi i k/m!}\{v\}).$$

Once we know they are well-defined, the limit in the definition makes it clear that the projection maps are functorial for S^1-equivariant continuous maps of Banach spaces.

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Lemma 1. *The projection maps p_n are well-defined.*

Proof. First we check that the limit converges. Using that the circle action is through isometries, together with the triangle inequality, we find that

$$\|\frac{1}{M!}\sum_{k=1}^{M!} e^{-2\pi ikn/M!}\cdot(e^{2\pi ik/M!}\{v\}) - \frac{1}{m!}\sum_{k=1}^{m!} e^{-2\pi ikn/m!}\cdot(e^{2\pi ik/m!}\{v\})\|_B \leq \tag{1.1}$$

$$\|\frac{m!}{M!}\sum_{k=1}^{M!/m!} e^{-2\pi ikn/M!}\cdot(e^{2\pi ik/M!}\{v\}) - v\|_B \leq \|\frac{m!}{M!}\sum_{k=1}^{M!/m!} (e^{-2\pi ikn/M!}\cdot(e^{2\pi ik/M!}\{v\}) - v)\|_B. \tag{1.2}$$

Since the circle action is continuous, given any $\varepsilon > 0$, there is some $m \gg 0$ such that

$$\|e^{2\pi iz}\{v\} - v\|_B < \varepsilon/2 \qquad \text{for } |z| < \frac{1}{m!},$$

and

$$|e^{-2\pi iz} - 1| < \varepsilon/2 \qquad \text{for } |z| < \frac{1}{m!},$$

so that we can (using that the circle action is through isometries) bound (1.1) by $\varepsilon \cdot \|v\|_B$ to see convergence.

Next, note that for any $q \in \mathbb{Q}$, $e^{2\pi iq}\{p_n(v)\} = e^{2\pi inq}\cdot p_n(v)$, since this equality actually holds with $\frac{1}{M!}\sum_{i=1}^{M!} e^{-2\pi ikn/M!}\cdot(e^{2\pi ik/M!}\{v\})$ replacing $p_n(v)$ for all $M \gg 0$ (how big M has to be depends on q). Since the action of the circle is continuous and $e^{2\pi iq}$ is dense in S^1 for $q \in \mathbb{Q}$, we must have $e^{iz}\{p_n(v)\} = e^{inz}\cdot p_n(v)$ for all $e^{iz} \in S^1$. □

The keen reader will note that the way we have defined the projection maps is essentially through Riemann sums, so that the projection p_n could be appropriately be rewritten

$$p_n(v) = \int_{S^1} e^{-inz}\cdot v(e^{iz})dz,$$

where we are considering v as a function from the circle via $v(e^{iz}) := e^{iz}\{v\}$. This formula closely resembles more classical definitions of Fourier coefficients, as we might hope. With our formulas in mind, we may now try to make question 1.1 more precise. A naive first guess is:

Question 1.2. *Let B be an S^1-Banach space. Then for any $v \in B$, is $v = \sum_{n\in\mathbb{Z}} p_n(v)$?*

Already with the space of continuous functions $C(S^1, \mathbb{C})$ we run into problems with the above formulation. Namely: there's no reason for the Fourier series as written to converge! Instead one typically works with Fejér sums to get an appropriate notion of a "sequence of Fourier series" converging to a given continuous function f.

Instead of asking that the Fourier series converge in a specific way, we can just ask that there is some notion of Fourier series which converges to any given vector, be it through something similar to Fejér sums, or even something more complicated. We now formulate the main question that this note aims to answer:

Question 1.3 (Generalized Fourier Series)**.** *Consider an S^1-Banach space B. Is the subspace $\bigoplus_{n\in\mathbb{Z}} B[n]$ of Fourier polynomials in B dense in the space B itself?*

Next section, we will find that the answer to this question is affirmative. The techniques used to arrive at this conclusion were inspired by Clausen-Scholze's lectures on analytic geometry.

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2. The Theorem

We begin with some preliminaries leading into our answer to Question 1.3.

Lemma 2. *If B is an S^1-Banach space, then there exists some set X, and an S^1-equivariant (isometric) embedding $B \to C(S^1 \times \beta X, \mathbb{C})$ into the space of continuous complex-valued functions on the product of the circle with the Stone-Čech compactification βX of X.*

Proof. Using the duality between Banach spaces and Smith spaces, our S^1-Banach space B dualizes to a Smith space M with a continuous circle action. Taking an appropriate compact subset of M as the set X (typically one will take for X the dual of the closed unit ball of B), there is a surjective map $\beta X \to X$ determined by the universal property of the Stone-Čech compactification. This gives rise to a cover $\mathcal{M}(\beta X, \mathbb{C}) \to M$ of M by the space of complex-valued Radon measures on βX. While no equivariance was enforced yet, we can replace βX by $S^1 \times \beta X$, to get an S^1-equivariant map $S^1 \times \beta X \to M$, extending to an S^1-equivariant map $\mathcal{M}(S^1 \times \beta X, \mathbb{C}) \to M$, which dualizes to the desired isometric embedding (the dualized map is isometric by the choice of X as the dual of the closed unit ball). □

This allows us to compare an arbitrary Banach space B to a somewhat more familiar space, in turn allowing us to prove:

Proposition 1. *Suppose that B is an S^1-Banach space such that $B[n] = 0$ for all $n \in \mathbb{Z}$. Then $B = 0$ is itself identically zero.*

Proof. Using Lemma 2, we find that, given an S^1-Banach space B, there is an isometric embedding $i : B \to C(S^1 \times \beta X, \mathbb{C})$ for some set X. By functoriality of the projection maps p_n, it suffices to show that for any set X, and any $f \in C(S^1 \times \beta X, \mathbb{C})$, $p_n(f) = 0$ for all n implies that f itself is zero. Note that $C(S^1 \times \beta X, \mathbb{C}) = C(\beta X, C(S^1, \mathbb{C}))$, so that any function in this space can be uniquely described as an X-indexed sequence $(f_x : S^1 \to \mathbb{C})_{x \in X}$ of complex-valued continuous functions on S^1.[1] The projection map p_n takes $f = (f_x)_{x\in X}$ to $p_n(f) = (p_n(f_x))_{x\in X}$. The claim now follows from the known fact that a nonvanishing continuous function on the circle has at least one nonzero Fourier coefficient (due to density of Fourier coefficients in $C^\infty(S^1, \mathbb{C})$ following from Fejér's Theorem, see this note, for example). □

Now we get to the main theorem of this note:

Theorem 1. *The answer to question 1.3 is positive. That is, if B is an S^1-Banach space, then the space $\bigoplus_n B[n]$ of generalized Fourier polynomials in B is a dense subspace.*

Proof. Take B to be any S^1-Banach space. Let V denote the closure of the subspace $\bigoplus_n B[n]$ in B, which itself inherits an action of the circle. As V is a closed sub-Banach space of B, we can take the quotient B/V, which is again a Banach space, inheriting a canonical continuous circle action through isometries. Let's write out the short exact sequence in order to give names to all the maps considered:

$$0 \longrightarrow V \overset{j}{\rightarrowtail} B \overset{q}{\twoheadrightarrow} B/V \longrightarrow 0.$$

[1]Subject to the extra condition that the closure of the set $\{f_x\}_{x\in X}$ in $C(S^1, \mathbb{C})$ is compact.

We claim that $(B/V)[n] = 0$ for all $n \in \mathbb{Z}$. Indeed, if there exists some $0 \neq v \in (B/V)[n]$, this lifts to a nonzero vector $v' \in B$. While v' may not live in $B[n]$, $p_n(v') \in B[n]$, and by functoriality of the projection maps, $q(p_n(v')) = p_n(q(v')) = p_n(v) = v$, as $v \in (B/V)[n]$. But $q(B[n]) = 0$, so $v = 0$ after all. It follows from Proposition 1 that $B/V = 0$, and we win. $\square$